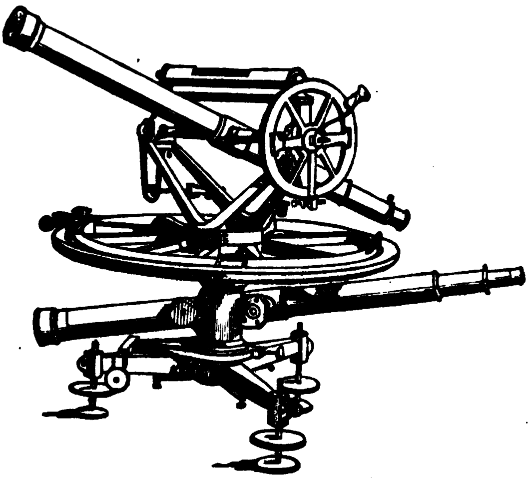
ELEMENTS

OF

GEOMETRY.



ELEMENTS

OF

GEOMETRY,

THEORETICAL AND PRACTICAL:

CONTAINING A FULL EXPLANATION

OF THE CONSTRUCTION AND USE OF TABLES,

AND

A NEW SYSTEM OF SURVEYING.

BY

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New-York:

PUBLISHED BY PRATT, WOODFORD, & CO.

1848.

Stereotyped by C. Davison & Co.,
33 Gold st., N. Y.

PREFACE.

WHILE some persons are complaining of constant innovation in text-books, and others finding fault equally with those in use, one scarcely knows whether or not to make any apology in putting forth a new work. One thing however, as it seems to me, is clear: that in view of the importance which is justly attached to elementary instruction, there can be little danger of too great a supply of manuals from which an enlightened community may select. If new books of geography and grammar, of arithmetic and algebra, are not only acceptable to the schools, in their onward march of improvement, but even indispensable for giving them life and vigor, why should objection be made to attempts in adapting the elements of geometry to the wants of the young and to the existing condition of instruction? Why should a blind veneration for antiquity cause the elements of Euclid to continue in one form or other in our schools, when the luminous Grecian himself, if now living, would, we doubt not, no longer employ them without a material remodeling in conformity with the mathematical methods of the day?

If boys are to learn that which they will practise when men, why should tyros be so long restricted to processes which, as mathematicians, they will seldom use?

Is it essential to the acquisition of competent skill in numbers that our arithmetics should be filled with examples, and to the comprehension of general principles that our algebras should observe, in the development of forms, an unbroken continuity of progression? why in the elegant science of geometry should there be neither example nor process? I am aware that these suggestions are not applicable, in all respects, to certain books on geometry recently published; but an elementary work of sufficient fullness, yet moderate in magnitude, highly practical, and, consequently, progressive in theory and example, is still, I believe, a *desideratum.* I have endeavored to prepare such a work; how well I have succeeded will, of course, be determined by others. Its chief feature

will be found to consist, not simply in the acquisition of geometrical principles, but in a regular *progression of method*, whereby it is intended to teach how to *geometrize.*

In pursuing quite out to the end of our geometrical studies, as well as at the beginning, the synthetic and undevelopable methods of the ancients, we acquire little or no power of going alone—we get some geometry, it is true, but still remain almost destitute of that education in analysis which is far more important. Why, in investigating the doctrines of forms, should we studiously keep out of sight the general principles of quantity, as if no such principles existed, when even Euclid himself could proceed but a little way without stopping to construct his, *to us*, clumsy book of proportion, the best and only algebra at his command?

Why, when so much labor is saved and greater clearness obtained, should we refuse to employ an equation like $(a+b)^2 = a^2+b^2+2ab$? Shall we have resources at hand and refuse to use them because Euclid was poor? Is it shorter, more satisfactory, or productive of finer results, to shut up the circumference of a circle between the perimeters of polygons than to avail ourselves of the simple symbol $\left[\frac{k}{h}\right]$, when employed as the vanishing ratio of the increments of two variables? Has geometry given birth to algebra, and shall she reap no advantage from her offspring? The succinct and methodical Francœur quoting Lagrange, says, "Tant que l'Algèbre et la Géométrie ont été séparées, leurs progrès ont été lents et leurs usages bornés; mais lorsque ces deux sciences se sont réunies, elles se sont prêté des forces mutuelles, et ont marché ensemble d' un pas rapide vers la perfection."* Again the luminous Lagrange, in the first of his "Leçons;" "Les fonctions dérivées se présentent naturellement dans la géométrie lorsq' on considère les aires, les tangentes,"&c.†

In accordance with these views, and in compliance with the recommendation of Lacroix, avoiding double methods, that we may be ever pressing on in that body of geometrical doctrines that are most useful, I have paid much attention to the classification, endeavor-

* So long as algebra and geometry were separated, their progress was slow and their application restricted; but when these two sciences became united, they lent each other mutual aid, and advanced together with rapid pace towards perfection.

† Derived functions present themselves naturally in geometry, when we consider areas, tangents, &c.

ing, consequently, to arrange the subjects in a natural order, so as to fall easily into families and readily develope each other. Thus, by the simple method of superposition, instead of a long, mixed, and circuitous route, the doctrines of parallels are presently arrived at, and, as a consequence, all the elementary theorems relating to angles and independent of the length of lines, are embraced in a first section of moderate length.

The comparison of equal figures follows; then that of proportional lines, which prepares the way for the investigation of areas—the doctrines of the circle are not considered till afterwards. Thus, in the first part, the topics are kept distinct, and, it is believed, in their natural order, by which means the progress is rendered more easy and rapid, and the methods of geometrizing are introduced, one after the other, as required by the gradually increasing difficulties of the growing subject.

For instance, in the first section little or no artifice is employed, and the simplest algebra, amounting to scarcely more than the common symbolical notation, is sparingly introduced in the second, while in the third the algebraic requisitions are somewhat increased, especially in the exercises. The method of incommensurables developed as a part of a system in the introductory book, is employed for the first time in the third section of the second, or first *geometrical Book;* the correlation of figures and change of algebraic signs find application in the more advanced propositions of the circle in the third Book; and the ratio of vanishing increments draws a tangent to the parabola in the close of the first Part. The elementary properties of the ellipse and parabola, being as simple as those of the circle and as useful in the study of natural philosophy and astronomy, are here introduced. Further, as proportion is generally included in our works on geometry, I have thought it advisable to insert an introductory Book, embracing, in a regular series of proportions the first doctrines of algebra, as being convenient for reference to those already acquainted with the science, and indispensable to others, who, by taking up these principles as required, may wish to proceed in the same class. The first Part, consisting of a hundred and twenty pages, is designed to embrace, in theory and practice, such an introductory body of elementary geometry—all the more difficult problems relating to perimeters or areas being postponed—as is required, not only to enter successfully upon the study of the higher investigations that follow, but for furnishing, in

some measure, with tangible and useful matter, those who want the disposition, lack the time, or have not the ability to proceed further. The first Book of the second Part consists of an elementary system of functions, depending on a single variable, and presented constantly under the simple notation. It embraces the binomial and logarithmic theorems. Every teacher must have observed that isolated methods, like those employed by Legendre in cases of incommensurability, make only such an impression upon the mind as to leave a sort of confusion always hanging about them, while that which forms part of a system readily commends itself to the understanding, and, consequently, remains ever after a permanent part of our appropriate knowledge. Laplace has well said, "Préférez, dans l' enseignement, les méthodes générales; attachez-vous à les présenter de la manière la plus simple, et vous verrez en même temps qu'elles sont toujours les plus faciles."* The method pursued in this book has been judged not only the most perfect† in itself, but, as will frequently happen when connected subjects, instead of being disjoined, are permitted to fall naturally together, at the same time the easiest. But aside from the indispensable matter which it contains, the chief object of this book is to prepare the way for what follows in the arithmetic of signs, the construction of trigonometrical tables, and the mensuration of surfaces and solids. In virtue of the course just alluded to, I have been enabled, in the second Book of the third Part, to make not only a more than usually full development of the trigonometrical forms with their application in the practical resolution of triangles, but to embrace also the quadrature of the circle and ellipse. In the next Book I have developed a system of surveying which I regard as peculiarly my own. It is true that the theorem for the computation of polygonal areas, which constitutes its chief feature, may be substantially found in Hutton, yet I have given to it so much of a new form and a demonstration at once general‡ and of the greatest simplicity, and extended it in so methodical a manner to the laying out and dividing of lands, that it becomes altogether another thing. Years of

* In instructing, adopt general methods; endeavor to present them in a manner the most simple, and you will see, at the same time, that they are the easiest.

† Not all the demonstrations in our algebras are perfect—for instance, the demonstrations of the binomial theorem in some school books, the most widely disseminated, amount to no demonstrations at all.

‡ The demonstration in Hutton is very tedious, and can hardly be said to be general.

instruction have proved, as hundreds of individuals would bear testimony, that the theorem here given will save, at the lowest estimate, two-thirds of the labor ordinarily incurred by the rectangular method. A further advantage is that, dispensing with a large and faulty table altogether, it is far more accurate—the computations being executed by aid of the common logarithmic numbers, calculated with greater care and usually extending to six or seven decimal places, and the operation being so ordered that, without any additional labor beyond what is absolutely essential to an honest confidence in the result, all gross errors, if any exist, whether of the field or the tables, are detected, and if these have no existence, the smaller and unavoidable ones very much reduced.

It is hoped that this book, often requested by my pupils, will prove acceptable to the schools generally.

Of the third Part, embracing the mensuration of solids, spherical trigonometry, and navigation, time will permit us to say little more than that, by the method pursued, we have been enabled, within moderate limits, to give a fuller development of these subjects than is usually found in our elementary books.

The modification and extension of Napier's Rules demands, how ever, a brief historic notice. I demonstrated and extended these rules by showing:

I. When $A = 90°$, $a_c = 90° - a$, $B_c = 90° - B$, $C_c = 90° - C$,

$$\left.\begin{aligned} \sin b &= \cos a_c \cos B_c = \tan C_c \tan c, \\ \sin c &= \cos a_c \cos C_c = \tan B_c \tan b, \\ \sin a_c &= \cos b \cos c \quad = \tan B_c \tan C_c, \\ \sin B_c &= \cos C_c \cos b \quad = \tan a_c \tan c, \\ \sin C_c &= \cos B_c \cos c \quad = \tan a_c \tan b : \end{aligned}\right\}$$

II. When $a = 90°$, $B_s = 180° - B$, $A_s = A - 90°$, &c.,

$$\left.\begin{aligned} \sin B_s &= \cos A_s \cos b_s = \tan c_s \tan C_s, \\ \sin C_s &= \cos A_s \cos c_s = \tan b_s \tan B_s, \\ \sin A_s &= \cos B_s \cos C_s = \tan b_s \tan c_s, \\ \sin b_s &= \cos c_c \cos B_s = \tan A_s \tan C_s, \\ \sin c_c &= \cos b_s \cos C_s = \tan A_s \tan B_s : \end{aligned}\right\}$$

III. When $c = a$, or the triangle is isosceles,

$$\left.\begin{aligned} \sin a_c &= \tan A_c \tan(\tfrac{1}{2}B)_c \\ \sin A_c &= \tan a_c \tan(\tfrac{1}{2}b), \\ \sin(\tfrac{1}{2}b) &= \cos a_c \cos(\tfrac{1}{2}B)_c, \\ \sin(\tfrac{1}{2}B) &= \cos A_c \cos(\tfrac{1}{2}b). \end{aligned}\right\}$$

Having shown the above extension to Mr. Dascum Green, then a pupil, he returned soon after, saying that he had not only verified my forms, but had obtained better ones, and presented the modifications of Napier's Rules as I had extended them, substantially as they will be found in the text.

This rule, as now extended and modified, possesses a greater simplicity and symmetry, and will enable us, in spherical astronomy, frequently to dispense with complicated figures.

I have added a small set of tables, extending to seven decimal places, calculated to answer the wants of the student while pursuing the work, and to make him more ready in using tabular numbers, by compelling him to interpolate by second differences. Afterwards he will find it decidedly to his advantage to possess himself of the tables recently published by Professor Stanley.

In conclusion, the advantages which we have endeavored to secure are:

1°. A better connected and more progressive method of geometrizing, calculated to enable the student to go alone.

2°. A fuller, more varied, and available practice, by the introduction of more than four hundred exercises, arithmetical, demonstrative, and algebraical, so chosen as to be serviceable rather than amusing, and so arranged as greatly to aid in the acquisition of the theory.

3°. The bringing together of such a body of geometrical knowledge, theoretical and practical, as every individual, laying any claim to a respectable education or entering into active life, demands.

4°. The furnishing to those who may wish to proceed on in mathematical learning, of a stepping-stone to higher and more extended works.

How well we have accomplished our object it is not, of course, for us to say. We have endeavored to render the work as mechanically correct as possible, but, residing at a distance from the place of publication, we can hardly expect that it will be entirely free from typographical imperfections.

G. C. W.

Genesee Wesleyan Seminary, June, 1848.

CONTENTS.

PART FIRST.

ALGEBRAICAL PRINCIPLES—PLANE GEOMETRY DEPENDING ON THE RIGHT LINE, AND ON THE CIRCLE, ELLIPSE, HYPERBOLA, AND PARABOLA.

BOOK FIRST.

ALGEBRAICAL PRINCIPLES.

SECTION I. *Use of the Signs—Fractions—Simple Equations.*

SECTION II. *Exponents—Proportion—Variation.*

Section III. *Analysis of Equations.*

BOOK SECOND.

PLANE GEOMETRY DEPENDING ON THE RIGHT LINE.

Section I. *Comparison of Angles.*

SECTION II. *Equal Polygons—First Relations of Lines and Angles.*

SECTION III. *Proportional Lines.*

SECTION IV. *Comparison of Plane Figures.*

BOOK THIRD.

PLANE GEOMETRY DEPENDING ON THE CIRCLE, ELLIPSE, HYPERBOLA, AND PARABOLA.

SECTION I. *The Circle.*

PART SECOND.

ANALYTICAL FUNCTIONS, PLANE TRIGONOMETRY, AND SURVEYING.

BOOK FIRST.

ANALYTICAL FUNCTIONS.

SECTION III. *General Laws relating to the Development of Functions depending on a single Variable.*

BOOK SECOND.

PLANE TRIGONOMETRY.

SECTION I. *Trigonometrical Analysis.*

SECTION II. *Resolution of Triangles, and Mensuration of Heights and Distances*

Section III. *Quadrature of the Circle, the Ellipse, and Parabola.*

BOOK THIRD.

SURVEYING.

Section I. *Description and Use of Instruments.*

Section II. *Plotting.*

Section III. *Computation of Areas.*

PART THIRD.

SOLID GEOMETRY, SPHERICAL GEOMETRY, AND NAVIGATION.

BOOK FIRST.

SOLID GEOMETRY.

SECTION I. *Planes.*

SECTION II. *Surfaces of Solids.*

SECTION III. *Volumes.*

BOOK SECOND.

SPHERICAL GEOMETRY.

SECTION I. *Spherical Trigonometry.*

BOOK THIRD.

NAVIGATION.

PART FIRST.

ALGEBRAICAL PRINCIPLES. — PLANE GEOMETRY DEPENDING ON THE RIGHT LINE, AND ON THE CIRCLE, ELLIPSE, HYPERBOLA, AND PARABOLA.

BOOK FIRST.

ALGEBRAICAL PRINCIPLES.

SECTION FIRST.

Use of the Signs—Fractions—Simple Equations.

Definition 1. Mathematics is a science, having for its object the investigation of the relations that quantities bear to each other.

Def. 2. We denominate Quantity that which admits of measurable increase or diminution.

Thus, lines, angles, weight and time, are quantities—but color, being incapable of a unit of measure, cannot, in the mathematical acceptation of the term, be regarded as a quantity.

Def. 3. A Proposition is anything proposed—if to be done, it is called a Problem; if to be demonstrated or proved, a Theorem; if a direct and necessary consequence of something going before, it is a Corollary; and if evident in itself, or not capable of being reduced to any simpler principle, it is known as an Axiom.

Explanation of the Signs. For the investigation of *general* propositions, it will often be desirable, and not unfrequently even indispensable, to be in possession of a method, equally general, whereby we may denote quantities and the operations to be performed upon them. The letters of the alphabet are employed for this purpose—the first commonly indicating known, and the last unknown quantities; but the student should accustom himself to regard any letter either as known or unknown. Instead of using different letters, the same differently marked, as a', read a prime; a'' [a second]; a_2 [a sub-two, or a second], and so on, are frequently introduced with advantage.

The symbols of operation for which we shall have more immediate use, are the signs of

Addition,	*Subtraction,*	*Multiplication,*	*Division,*
$+$ (plus, more)	— (minus, less)	$\times$ or $\cdot$	: or $\div$

Aggregation,	*Equality,*	*Inequality,*	*Deduction,*	*Continuation.*
$()$, $[()]$, $\{[()]\}$	$=$	$>$ (greater), $<$	$\therefore$	$\ldots$

Thus, that 4 is to be added to 6, is written, $6+4$, and read, *six plus four;* that six and four are equal to 12 diminished by two, is written,

$$6+4=12-2,$$

and read, 6 *plus* 4 is equal to 12 *minus* 2; that the sum of 6 and 4 multiplied by 2, is equal to 44 diminished by 4, and the remainder divided by two, is written,

$$(6+4)\times 2=(44-4)\div 2,$$

or $$(6+4)\cdot 2=(44-4):2,$$

or $$2(6+4)=(44-4):2;$$

for when the omission of a sign, as is done in $2(6+4)$, would not be attended by any ambiguty, it may be dropped. Thus $a\,b$ signifies that a is to be multiplied into b. Propositions are much shorter in symbolical than in common language, and are, consequently, more clearly expressed, as has already been shown, and as will appear in a still stronger light by writing an example or two in corresponding columns.

	$a=b,$	the	a is equal to b,
	$c=b;$	same	and c is equal to b;
$\therefore$	$a=c.$	as	therefore a is equal to c.

$\frac{(a+b)(a-b)}{m}$ $=\frac{aa-bb}{m}$ $<\frac{aa}{m}$	The sum of a and b multiplied into the difference of a and b, and this product divided by m, gives a quotient which is equal to the quotient arising from dividing the remainder of the square of a diminished by the square of b, by m, which again is less than the quotient of the square

of a divided by m.

Scholium. The student should be accustomed to turn the algebraical into common and appropriate language; thus the sign of equality, $=$, will commonly be read, "is equal to," but sometimes, "will be equal to," and at others, "equalling;" the symbol of deduction, $\therefore$, will generally be read, "therefore," sometimes, "hence," "it follows," and again, "from what goes before, we infer," &c.; the symbol of continuation, consisting of three points,

..., will be enunciated, "&c.," "and so on," "continued according to the same law."

PROPOSITION I. [AXIOM.]

The whole is equal to the sum of all its parts. (1)

This proposition is an axiom, that is, evident of itself; no words about it, therefore, can make it any plainer.

Corollary 1. The whole is greater than any of its parts, (1_2) or, the whole exceeds any of its parts by those which are excepted—otherwise the whole would differ from the sum of all its parts.

Cor. 2. Quantities which are equal in all their parts, are (1_3) said to be equal to each other; for the whole is known by its parts —or, quantities which are not evidently identical, can be compared only by a resolution into like or unlike parts.

Cor. 3. Quantities which, however resolved, are unequal (1_4) in any of their parts, are not equal to each other (1_3).

Cor. 4. Quantities which are equal to each other, are (1_5) equal in all their parts; for, if some of their parts were unequal, they would, by (1_4), be themselves unequal; ∴

Cor. 5. Unequal quantities are not equal in all their parts. (1_6)

Cor. 6. Quantities which are equal to the same or equal (1_7) quantities, are equal to each other; for they are equal in all their parts, (1), (1_5).

Cor. 7. Quantities measured by the same or equal quantities, (1_8) are equal to each other; for equality of measures implies equality of parts, whether the measuring quantities be of the same kind with those measured or not. Thus, two masses of lead are equal in weight when they both contain the same number of pounds, or when they both contain the same number of cubic inches.

Cor. 8. Quantities are to each other as their measures; ∴ (1_9)

Cor. 9. Of quantities having unequal measures, that is (1_{10}) the greater to which the greater measure belongs.

Cor. 10. If the same or equal quantities be increased or (1_{11}) diminished by the same or equal quantities, the resulting quantities will be equal to each other; since they will be equal in all their parts, (1_5), (1).

Cor. 11. If the same or equal quantities be multiplied or (1_{12}) divided by the same or equal quantities, the resulting quantities

will be equal; since multiplication is repeated addition, and division a continued subtraction.

Cor. 12. If equal quantities be raised to the same powers, or the same roots be taken of them, the resulting quantities will be equal; since a power is formed by continued multiplication, and a root is extracted by the converse operation. (1_{13})

Cor. 13. The same or equal quantities, by the same or equivalent operations, give the same or equal quantities. (1_{14})

Cor. 14. Quantities satisfying the same or equivalent conditions, are equal to each other. (1_{15})

Cor. 15. Unequal quantities, by the same or equivalent operations, will continue to be unequal, and in the same sense—that is, the greater will be the greater still (1_{14}). See also (1_{11}), (1_{12}), (1_{13}). Thus, if a be greater than b,[$a > b$], a increased by c will be greater than b increased by c [$a+c, > b+c$], a diminished by c will be greater than b diminished by c,[$a—c > b—c$], m times a will be greater than m times b,[$ma > mb$], &c. (1_{16})

Cor. 16. If inequalities, taken in the same sense, be added, the result will be an inequality also in the same sense (1_{16}). Thus, if a be $> b$ and $c > d$, then $a+c > b+d$. (1_{17})

Cor. 17. When inequalities, whose differences are the same, are added in a contrary sense, the result will be an equality (1_3). Thus, if a be as much $> b$ as c is $< d$, then $a+c$ will $= b+d$. (1_{18})

Cor. 18. When inequalities are added in a contrary sense, the sense of the resulting inequality will be that of the greater (1_{18}). Thus, if $a > b$ and $c < d$, then will $a+c > b+d$, provided the difference between a and b be greater than the difference between c and d,[$a-b > d-c$, $\therefore a > d-c+b$. $a+c > b+d$]. (1_{19})

Cor. 19. If inequalities, taken in the same sense, be subtracted, the one from the other, the resulting inequality will be in the same or a contrary sense, according as the minuend is the greater or less inequality. (1_{20})

PROPOSITION II. [COROLLARY FROM 1_8.]

Magnitudes which may be made to coincide throughout, are equal to each other. (2)

The magnitudes are equal in all their parts.

Cor. 1. When one magnitude embraces another without being filled by it, the first is greater than the second (1_2). (2_2)

Cor. 2. Magnitudes measured by the same or equal magnitudes, are equal to each other (1_8). (2_3)

Cor. 3. Magnitudes are to each other as their measures (1_9). (2_4)

Cor. 4. Of magnitudes having unequal measures, that possessing the greater measure is the greater. (2_5)

Scholium I. It is sometimes convenient to make a distinction between *equal* and *equivalent*, but the terms will generally be used as synonymous.

Scholium II. It is obvious that all propositions requiring demonstration, must be founded, either directly or indirectly, upon those which do not, or on axioms; and hence our first proposition becomes the source of a vast amount of knowledge.

Def. 4. A coëfficient is a figure employed to show how many times a letter is taken; thus, in $3a$, 3 is the coëfficient of a, and $3a = a + a + a$. A letter may be regarded as a coëfficient, as n in

$$na = a + a + a + \dots [n \text{ times}].$$

Def. 5. Operations are said to be *relatively free* when the result is the same in whatever order they are executed, the one after the other.

Thus, O, O_2, the two parts of a compound operation, are relatively free when

$$O_1\ (O_2) = O_2\ (O_1).$$

PROPOSITION III.

*Additions, subtractions, additions and subtractions, are relatively free operations—*that is, *the terms of a polynomial may be inverted at pleasure.* (3)

1°. *Additions.* That $4+3$ is $=$ to $3+4$ will be evident from counting the units into which the two sums are resolvable;

thus $\quad 4+3=(1+1+1+1)+(1+1+1)$,

and $\quad 3+4=(1+1+1)+(1+1+1+1)$;

$\therefore (1_3) \quad 4+3=3+4.$

So $a+b=(1+1+1+\dots [a \text{ units}])+(1+1+1+\dots [b \text{ units}])$

$\quad = 1+1+1+\dots [a+b]$

$\quad = (1+1+1+\dots [b])+(1+1+1+\dots [a]) = b+a,$

which was to be proved.

2°. *Subtractions.* The remainder arising from diminishing a units first by b units and then by c units will be found the same as that

obtained by diminishing a first by c and then by b, or $a-b-c=a-c-b$; for, let a contain $r+b+c$ units, or

$a=r+b+c$, which $(1^\circ)=r+c+b$;

then (1_{11}) $a-b=r+c$, subtracting b from both sides,

and $a-b-c=r$, subtracting c from the last equation—

or $a-c=r+b$, subtracting c from the first,

and $a-c-b=r$, subtracting b from the last;

$\therefore (1_7)$ $a-b-c=r=a-c-b$, Q. E. D.*

3°. *Additions and subtractions.* $a+b-c=a-c+b$; for we obviously have the same number of units whether we diminish the $a+b$ units by taking the c units from b or from a. Q. E. D.

PROPOSITION IV.

Multiplications, divisions, multiplications and divisions, are relatively free operations. (4)

1°. *Multiplications.* We may know that $3\cdot4=4\cdot3$ by resolving the numbers into their component units, and setting these down in an orderly way to count;

thus, by counting, we find 4 units repeated 3 times, the same as 3 units repeated 4 times, or $3\cdot4=4\cdot3$.

$$3\text{ units}\left\{\overbrace{\begin{array}{cccc}1&1&1&1\\1&1&1&1\\1&1&1&1\end{array}}^{4\text{ units.}}\right.$$

So, a units $(=1+1+1+\dots[a])$ repeated b times, is the same as b units $(=1+1+1+\dots[b])$, repeated a times—

or $ab=ba$;

$$b\text{ units.}\left\{\overbrace{\begin{array}{cccc}1&1&1&\dots\\1&1&1&\dots\\1&1&1&\dots\\&\dots&&\dots\\&\dots&&\dots\\&\dots&&\dots\end{array}}^{a\text{ units.}}\right.$$

hence (1_{12}) a times b units, repeated c times will be equal to b times a units repeated c times, but b times a units, repeated c times, by what has just been demonstrated, is $=$ to c units repeated b times a times—and in order to repeat c units b times a times, we may multiply first by b and then by a; for, multiplying by b instead of ba, is multiplying by a number a times too small, and consequently, the product, being a times too small, will be corrected by multiplying again by a; all which may be set down in symbols thus;—

* "*Quod erat demonstrandum,*" which was to be proved.

$$ab = ba,$$

$\therefore$ (1_{12}) $\qquad (ab)c = (ba)c = c(ba) = cba,$

or $\qquad abc = bac = cba,$

where it will be observed that a is made to occupy every place in the product, and in like manner the same may be shown of b and c —and the reasoning may be extended by introducing additional factors at pleasure. Q. E. D, for 1°.

Cor. 1. The factors of a product may be grouped in multiplication at pleasure. (4_2)

Cor. 2. Any factor may be regarded as the coëfficient to the remaining factors of the product (4_2). (4_3)

2°. *Division.* Assuming any quantity, Q, to be divisible by others, as b and c, is obviously the same as assuming Q to be resolvable into factors, two of which are b and c; hence, Q being divisible by b and c, denoting the third factor by a, we have

$$Q = abc;$$

$\therefore$ (1_{12}) $\qquad Q : c = ab$, dividing both sides by c, and observing that the product abc is divided by c by omitting the multiplier c; $\therefore$ dividing the last by b, there results

$$(Q : c) : b = a.$$

But (1°) $\qquad Q = abc = acb;$

$\therefore$ $\qquad Q : b = ac,$

and $\qquad (Q : b) : c = a;$

$\therefore$ (1_7) $\qquad (Q : b) : c = (Q : c) : b.$ Q. E. D. for 2°.

3°. *Multiplication and Division.*—Let Q be divisible by b, or

$$Q = ab;$$

$\therefore$ $\qquad Q : b = a,$

and, multiplying by c, $(Q : b) \cdot c = ac;$

again $\qquad Q \cdot c = ab \cdot c = acb,$

$\therefore$ $\qquad (Q \cdot c) : b = ac;$

hence $\qquad (Q : b) \cdot c = (Q \cdot c) : b.$ Q. E. D. for 3°, and the proposition is proved.

PROPOSITION V.

Additions and Subtractions in regard to multiplications and divisions, are relatively free. (5)

1°. Quantities otherwise alike are added and subtracted by adding and subtracting their coëfficients. (5_2)

Thus, 3 times 4 and 2 times 4 are obviously $3+2$ or 5 times 4.

$$3 \cdot 4 + 2 \cdot 4 = (3+2) \cdot 4 = 5 \cdot 4.$$

$$4\left\{\begin{matrix} \overbrace{1 \quad 1 \quad 1}^{3} & \overbrace{1 \quad 1}^{+\quad 2} \\ 1 \quad 1 \quad 1 & 1 \quad 1 \\ 1 \quad 1 \quad 1 & 1 \quad 1 \\ 1 \quad 1 \quad 1 & 1 \quad 1 \end{matrix}\right.$$

So a times c plus b times c is equal to $a+b$ times c, or $ac+bc=(a+b)c$;

$\therefore$ $(a+b)c=ac+bc$, inverting the members of the equality, which obviously may be done, since it is only asserting the same proposition in an inverted order;

$$c\left\{\begin{matrix} \overbrace{1+1+1+\ldots}^{a} & + & \overbrace{1+1+1+\ldots}^{b} \\ 1+1+1+\ldots & + & 1+1+1+\ldots \\ 1+1+1+\ldots & + & 1+1+1+\ldots \\ \cdot \quad \cdot \quad \cdot \quad \ldots & & \cdot \quad \cdot \quad \cdot \quad \ldots \\ \cdot \quad \cdot \quad \cdot \quad \ldots & & \cdot \quad \cdot \quad \cdot \quad \ldots \\ \cdot \quad \cdot \quad \cdot \quad \ldots & & \cdot \quad \cdot \quad \cdot \quad \ldots \end{matrix}\right.$$

$\therefore$ (1_{11}) $(a+b)c-bc=ac$, subtracting bc from both sides; but, as a and b may be any quantities whatever, $a+b$ may be any whatever, and we may substitute m for $a+b$, n for b, and, consequently, $m-n$ for $(a+b)-b=a$;

$$\left.\begin{array}{ll} \text{whence} & mc-nc=(m-n)c, \\ \text{or} & (m-n)c=mc-nc. \end{array}\right\} \qquad (5_2)$$

Scholium. The last form, resulting from the hypothesis (5_4) that m is greater than n $[(a+b)>b]$, must also be adopted when m is less than n; otherwise general symbols for the representation of quantities would have to be abandoned altogether, as it will frequently be impossible, as well as generally inconvenient, to distinguish between m and n, whether m be greater or less than n. And in order to thus extend the application of the form, or to make it general, we have only to interpret the expressions $mc-nc$, $(m-n)c$, both when m is greater and less than n: 1°, when $m>n$, we have $mc>nc$ and, consequently, $mc-nc$, $m-n$, both plus; but 2°, when $m<n$, we find $mc<nc$, and $mc-nc$, $m-n$, both become minus.

It is easy now to extend the operation to any number of terms, whether plus or minus.

$$\begin{aligned} \text{Thus,}\quad ax+bx-cx+dx+\ldots &= (a+b)x-cx+dx+\ldots \\ &= (a+b-c)x+dx+\ldots \\ &= (a+b-c+d)x+\ldots \\ &= (a+b-c+d+\ldots)x. \end{aligned}$$

So the first part of the proposition is proved.

2°. Dividing both sides (1_{12}) of the last equation by x, we have $(ax+bx-cx+dx+...):x=a+b-c+d+...$

$$=(ax):x+(bx):x-(cx):x+(dx):x+...,$$

and the proposition is proved.

Cor. 1. A *Polynomial*, or algebraical expression consisting of many terms, is multiplied or divided by multiplying or dividing its terms. (5_5)

Cor. 2. One polynomial is multiplied into another by multiplying all the terms of the one into all the terms of the other. (5_6)

Thus, $(a+b+c+...)(a_2+b_2+c_2+...)=(a+b+c+...)a_2 +(a+b+c+...)b_2+(a+b+c+...)c_2+...=aa_2+ba_2+ca_2 +...+ab_2+bb_2+cb_2+...+ac_2+bc_2+cc_2+...$ (5_6).

Cor. 3. The number of terms in a polynomial product, is equal to the product of the numbers denoting the terms in the constituent polynomial factors. (5_7)

Thus, if P_a, P_b, P_c, ... $[m]$, denote polynomials of a, b, c, ... terms, the polynomials being m in number, we have

$P_a \cdot P_b = P_{ab}$, a polynomial of a times b terms;

$\therefore$ $P_a \cdot P_b \cdot P_c = P_{ab} \times P_c = P_{abc}$,

and generally $P_a \cdot P_b \cdot P_c \cdot ... [m] = Pabc \cdot ... [m]$. (5_8)

If we make the number of terms the same, a for instance, in all the polynomial factors, or put $a=b=c=...$, there results,

$$P_a \cdot P_a \cdot P_a \cdot ... [m] = Paaa ... [m],$$

or $(P_a)^m = Pa^m$, i. e., (5_9)

Cor. 4. The number of terms in the mth power of a polynomial of a terms, is equal to the mth power of a. Thus, the number of terms in the expansion of the sixth power of the binomial $x+y$, $[(x+y)^6]$ will be found to be $2^6=64$.

Def. 6. Operations like those preceding, which may be performed upon the whole of a polynomial at once, or upon its parts separately, are denominated *linear*.

Cor. 5. The compounds of the above linear operations, are themselves linear. (5_{10})

Thus, if we *multiply* any polynomial, $x+y+z+...$, by any quantity a, and then *divide* by b, we find

$$[a(x+y+z+...)]:b=(ax):b+(ay):b+(az):b+....$$

PROPOSITION VI.

A product made up by the multiplication of additive quantities, is itself additive; and is changed in sign by changing the sign of any one of its factors. (6)

This proposition will become evident by comparing (5_3) and its extension in (5_4) with the first part of (4); for, observing that $m - n$ may represent any quantity either plus or minus, by making m greater or less than n by that quantity, and that $m - n$, $mc - nc$ are both plus when m is greater than n, [$m > n$], and both minus when $m < n$, $(m - n)\ c = mc - nc$, becomes $+ (m - n) \cdot + c = + (mc - nc)$, or $+ \cdot + = +$ when $m > n$, and $- (m - n) \cdot + c = - (mc - nc)$, or $- \cdot + = -$, when $m < n$; but the order of the factors may be changed at pleasure, $c\ (m - n) = (m - n)\ c$, $\therefore + \cdot - = - \cdot + = -$: from all which it follows that changing the sign of a factor changes the sign of the product into which it enters.

Thus $\quad + x \cdot + y \cdot + z \cdot ... = + (xy) \cdot + z \cdot ... = + (xyz) \cdot ...$
$= + xyz \cdot ...,$

and $\quad - x \cdot + y \cdot + z \cdot ... = - (xy) \cdot + z \cdot ... = - (xyz) \cdot ...$
$= - xyz \cdot ...;$

$\therefore \quad - x \cdot - y \cdot + z \cdot ... = + xyz \cdot ...,$

and $\quad - x \cdot - y \cdot - z \cdot ... = - xyz \cdot ...$, &c., &c.

Cor. 1. An even number of minus factors gives a plus product—an odd number, a minus product. (6_2)

Thus, $- \cdot - \cdot - \cdot - \cdot ... [2n]^* = +, - \cdot - \cdot - \cdot ... [2n + 1] = -.$

Cor. 2. An even power of a plus or minus quantity is plus (6_2). Thus $(+ a)^2 = + a^2$, $(- a)^2 = - a \cdot - a = + a^2$, $(\pm a)^{2n} = + a^{2n}$. (6_3)

Cor. 3. An even root of a plus quantity is either plus or minus [$\pm$], (6_3). Thus $\sqrt[2n]{+ a^{2n}} = \pm a$. ($6_4$)

Cor. 4. An even root of a minus quantity is *imaginary*, that is, impossible; for (6_4) it can be neither plus nor minus. (6_5)

Thus, in $\sqrt[2n]{- a} = \pm r$, r is imaginary; for, raising to the ($2n$)th power, we have $- a = (\sqrt[2n]{- a})^{2n} = (\pm r)^{2n}, = + r^{2n}, - = +$ impossible.

Cor. 5. An odd power of a minus quantity is minus (6_2). Thus, $(- x)^3 = - x \cdot - x \cdot - x = - x^3$, $(- x)^{2n+1} = - x^{2n+1}$. ($6_6$)

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* Whatever n may be, $2n$ is an even number.

Cor. 6. An odd root of a minus quantity is minus (6_6). (6_7)
Thus, $\sqrt[2n\pm1]{-a}=-r$; for $-a=(-r)^{2n\pm1}=-r^{2n\pm1}$, $-=-$.

Cor. 7. In products and quotients, like signs [+, +, or −, −] (6_8) produce plus [+] and unlike signs [+, −, or −, +,], minus [−].

For products this has already been shown (6_2), and, to establish the same for quotients, let q be the quotient arising from dividing any quantity D by any other d.

or $\quad D:d=q$,

whence (1_{12}) $\quad (D:d)\cdot d=qd$, multiplying both sides by d,

or (4) $\quad (D\cdot d):d=D=qd$;

$$\therefore \quad \begin{aligned} +D&=+q\cdot+d, \text{ whence } +:+=+,\\ -D&=+q\cdot-d, \quad \ldots \quad -:-=+,\\ +D&=-q\cdot-d, \quad \ldots \quad +:-=-,\\ -D&=-q\cdot+d, \quad \ldots \quad -:+=-. \end{aligned}$$ Q. E. D.

Cor. 8. In products and quotients the signs +, −, are (6_9) relatively free (6_8). Thus, $+\cdot-=-\cdot+$, $+:-=-:+$.

PROPOSITION VII.

The square of a polynomial is equal to the sum of the (7) *squares of the terms and their double products taken two and two.*

For in (5_6) making $a_2=a$, $b_2=b$, $c_2=c$, ... , and arranging, we have

$$(a+b+c+\ldots)^2=a^2+b^2+c^2+\ldots+2ab+2ac+\ldots+2bc+\ldots \qquad (7)$$

Cor. 1. The square of a binomial is equal to the sum of (8) the squares of its terms increased by their double product. For in (7) making all the letters nothing except a and b, there results

$$(a+b)^2=a^2+b^2+2ab=a^2+2ab+b^2=a^2+(2a+b)\,b. \qquad (8)$$

Cor. 2. The square of a residual is equal to the sum of the (9) squares of the terms diminished by their double product.

For changing the sign of b in (8) we have

$$(a-b)^2=a^2+(-b)^2+2a\,(-b),$$

or (6_3), (6), $\quad (a-b)^2=a^2+b^2-2ab. \qquad (9)$

PROPOSITION VIII.

The product of the sum and difference of two quantities (10) *is equal to the difference of their squares, and vice versâ, the difference of the squares of two quantities is equal to the product of their sum and difference.*

For in (5_6) making all the letters nothing except a, b, a_2, b_2, and changing a_2 into a, b_2 into $-b$, we get

$$\left.\begin{aligned}(a+b)(a-b) &= a^2-b^2,\\ \therefore\quad a^2-b^2 &= (a+b)(a-b).\end{aligned}\right\} \qquad (10)$$

Q. What relation does (5_6) bear to (7)? (7) to (8)? (8) to (9)? (5_6) to (10)? Which is the more general (5_6) or (7)?

Scholium. Forms (7), (8), (9) and (10), are important for their applications, and remarkable for illustrating the facility with which general truths are discovered by the employment of algebraical language.

PROPOSITION IX.

A polynomial may be freed from the minus sign, or on the contrary subjected to its influence, by changing the signs of all its terms. (11)

Every polynomial, as $3a-5b+c+2d-e$, may be, so far as the signs are concerned, reduced to the form $+B-C$, representing the sum of the plus terms, as $3a+c+2d$, by B, and the sum of the minus terms, as $-5b-e$, by C. Therefore all cases of subtraction will be comprised in this general form,

$$A-(+B-C),$$

where it is required to take the polynomial $(+B-C)$ from any quantity A. Now if from A we subtract B, the sum of the plus terms, we have $A-B$, by which all the terms in B originally $+$ become $-$; but, in taking the *whole* of B from A, we have diminished A by a part of B, namely C, which ought not to have been subtracted, since the true subtrahend, $B-C$, is only that part of B remaining after the diminution of B by C; therefore, $A-B$ being too small by C, the true remainder becomes $A-B+C$, on the addition of C—and all the terms embraced in $(+B-C)$ change their signs in $(-B+C)$, also

$$A-(+B-C)=A-B+C,$$

or

$$-(+B-C)=-B+C,$$

or

$$-B+C=-(+B-C.) \qquad \text{Q. E. D.}$$

Remark. The student should be familiar with the resolution of polynomials into factors, not only by the addition and subtraction of coëfficients, but by the employment of the theorems under (7), (8), (9), (10).

EXERCISES.

1°. What is the sum of fifteen times x and seventeen times x? $15x+17x=$ how many times x?

2°. $ax+bx=$? 3°. $2(a+b)+3(a+b)=$? 4°. $a(c+d)+b(c+d)=$?

5°. $10b-7b=$? 6°. $35b-20b=$? 7°. $ac-bc=$?

8°. $-ad+bd=-(ad-bd)=$? 9°. $mx+nx-rx=$?

10°. $ac-ad+bd-bc=(a-b)(c-d,)$, how?

11°. Resolve a^2-b^2, a^4-b^4, a^8-b^8, a^6-b^6, $a^{10}-b^{10}$, $a^{2n}-b^{2n}$, into their simplest factors.

12°. Resolve x^2+2x+1, x^2+1-2x, $1-2(xy-x+y)+xx+yy$, into their prime factors.

PROPOSITION X.

The degree of a product is equal to the sum of the numbers indicating severally the degrees of its factors. (12)

For, as two factors, ab, multiplied by a third, a_2, give a product of three factors, aba_2, so a monomial of p factors $abc\ldots[p]$, multiplied by an additional factor, a_2, gives a product of $p+1$ factors, $abc\ldots\times a_2\,[p+1]$, and by a second factor, $abc\ldots\times a_2b_2\,[p+2]$; and generally, $abc\ldots[p]\times a_2b_2c_2\ldots[q]=abc\ldots\times a_2b_2c_2\ldots[p+q]$;

$\therefore\quad abc\ldots[p]\times a_2b_2c_2\ldots[q]\times a_3b_3c_3\ldots[r]=abc\ldots\times a_2b_2c_2\ldots\times a_3b_3c_3\ldots[p+q+r]$;

$\therefore$ in general, $abc\ldots[p]\times a_2b_2c_2\ldots[q]\times a_3b_3c_3\ldots[r]\times\ldots=abc\ldots\times a_2b_2c_2\ldots\times a_3b_3c_3\ldots[p+q+r+\ldots]$. (12)

Cor. 1. Powers of the same letter are multiplied together by adding their exponents. (13)

For from (12), making b, c, ... a_2, b_2, c_2, ... a_3, b_3, c_3, ... all $=a$, there results,

$$\left.\begin{array}{l} aaa\ldots[p]\times aaa\ldots[q]\times aaa\ldots[r]\times\ldots=aaa\ldots[p+q+r+\ldots], \\ \text{or}\qquad a^p\cdot a^q\cdot a^r\cdot\ldots=a^{p+q+r+\ldots}. \end{array}\right\}\quad(13)$$

Cor. 2. A quantity is involved by multiplying its exponent by the index of the power to which it is to be raised. (14)

For, making $p, q, r, \ldots$ all $=m$ and n in number (13), becomes

$$\left.\begin{array}{l} a^m\cdot a^m\cdot a^m\cdot\ldots[n]=a^{m+m+m+\ldots[n]}, \\ (a^m)^n=a^{mn}. \end{array}\right\}\quad(14)$$

Cor. 3. Powers of the same letter are divided by diminishing the exponent of the dividend by that of the divisor. (15)

For, from (13) we have

$$\left.\begin{array}{ll} & a^{p+q} = a^p \cdot a^q\,; \\ \therefore & a^{p+q} : a^q = a^p = a^{(p+q)-q} \\ \text{or } [p+q=x,\ q=y], & a^x : a^y = a^{x-y}. \end{array}\right\} \qquad (15)$$

Cor. 4. When one polynomial is divided by another, the highest power of any letter in the dividend divided by the highest power of the same letter in the divisor, gives the highest power of that letter in the quotient. (16)

For (15), *divisor* $a^q \times$ *quotient* $a^p =$ *dividend* a^{p+q}.

Scholium. It will be found convenient in division to arrange (3) the polynomials according to the descending powers of a given letter.

EXERCISES.

1°. Divide $a-b$ by $a-b$. 2°. Divide a^2-b^2 by $a-b$. 3°. Divide a^3-b^3 by $a-b$. 4°. Divide a^4-b^4 by $a-b$.

Operation.

$$\begin{array}{l} a-b)\ a^4 - b^4(a^3 + a^2b + ab^2 + b^3 \\ \quad\ \ \underline{a^4 - a^3b} \\ \qquad\ \ +a^3b \\ \qquad\ \ \underline{+a^3b - a^2b^2} \\ \qquad\qquad\ \ +a^2b^2 \\ \qquad\qquad\ \ \underline{+a^2b^2 - ab^3} \\ \qquad\qquad\qquad\ \ +ab^3 - b^4 \\ \qquad\qquad\qquad\ \ \underline{+ab^3 - b^4} \end{array}$$

The division is here executed by the powers of a (16).

Subtract by changing the sign (11).

5°. Divide a^5-b^5, a^6-b^6, a^7-b^7, ..., a^n-b^n, by $a-b$.

PROPOSITION XI.

1°. *Multiplying the dividend while the divisor remains the same, or dividing the divisor while the dividend remains the same, multiplies the quotient;* (17)

2°. *Dividing the dividend while the divisor remains the same, or multiplying the divisor while the dividend remains the same, divides the quotient;* $\therefore$

3°. *The value of the quotient is not altered by multiplying or dividing both divisor and dividend by the same quantity.*

Def. 7. A *Fraction* is an expression of division, and arises from an impossibility of performing the operation. Therefore, indicating the dividend, now called the *numerator*, by N, the divisor or *denominator* by D, and the quotient or *value* of the fraction by V, we have

$$N : D = V, \text{ or } N = DV; \text{ hence (17)},$$

Cor. 1. A fraction is multiplied by multiplying its numerator, N, or by dividing its denominator, D, $[N \cdot R = D \cdot VR, \&c.]$. (18)

Cor. 2. A fraction is divided by dividing its numerator, or by multiplying its denominator. (19)

Cor. 3. A fraction is changed in form, without being changed in value, by multiplying or dividing both numerator and denominator by the same quantity. (20)

Cor. 4. If a fraction be multiplied by its denominator, the product will be the numerator. (21)

For (18) the fraction $\frac{n}{d}$ multiplied by $d = \frac{nd}{d} = (nd) : d = n$.

Cor. 5. Fractions are multiplied together by taking the product of their numerators for a new numerator, and that of their denominators for a new denominator. (22)

$$\text{For } \frac{N}{D} \text{ times any quantity } Q = Q \text{ times } \frac{N}{D} \text{ (4)}, = \frac{QN}{D} \text{ (18)};$$

$$\therefore \quad \text{substituting } \frac{N_2}{D_2} \text{ for } Q, \ \frac{N}{D} \text{ times } \frac{N_2}{D_2} = \frac{\frac{N_2}{D_2} \cdot N}{D} = \frac{\frac{NN_2}{D_2}}{D} \text{ (18)},$$

$$= \frac{NN_2}{DD_2} \text{ (21)},$$

multiplying both numerator and denominator by D_2 (20);

$$\therefore \quad \frac{N}{D} \cdot \frac{N_2}{D_2} \cdot \frac{N_3}{D_3} \cdot \ldots = \frac{NN_2}{DD_2} \cdot \frac{N_3}{D_3} \cdot \ldots = \frac{NN_2N_3}{DD_2D_3} \cdot \ldots \qquad (22)$$

Cor. 6. A fraction may be raised to any power by involving its numerator and denominator separately, and consequently, any root of a fraction will be found by an evolution operated upon its terms. (23)

For, making $N_2, N_3, \ldots = N$,

and $D_2, D_3, \ldots = D$, (22) becomes

$$\frac{N}{D} \cdot \frac{N}{D} \cdot \frac{N}{D} \cdot \ldots [m] = \frac{NNN \ldots [m]}{DDD \ldots [m]}, \text{ or } \left(\frac{N}{D}\right)^m = \frac{N^m}{D^m}. \qquad (23)$$

Cor. 7. To divide by a fraction, invert it, and proceed as in multiplication. (24)

Let $Q : \frac{N}{D}$ be any quantity divided by a fraction ; multiplying both dividend and divisor by $\frac{D}{N}$ (17), we have

$$Q : \frac{N}{D} = \left(Q \cdot \frac{D}{N}\right) : \left(\frac{N}{D} \cdot \frac{D}{N}\right) = \left(Q \cdot \frac{D}{N}\right) : \frac{ND}{ND} = \left(Q \cdot \frac{D}{N}\right) : 1$$

$$= Q \cdot \frac{D}{N} \quad \text{Q. E. D.}$$

Cor. 8. The reciprocal of a fraction is the fraction inverted. (25)

For $Q : \frac{N}{D} = Q \cdot \frac{D}{N}$, becomes $1 : \frac{N}{D} = 1 \cdot \frac{D}{N} = \frac{D}{N}$, when $Q = 1$.

Cor. 9. Any quantity, whether whole or fractional, will be reduced to a given denominator, by being multiplied by this denominator and set over it. (26)

For we have $$Q = Q \cdot 1 = Q \cdot \frac{D}{D} = \frac{QD}{D}.$$

Thus, $\frac{1}{2} = \frac{\frac{1}{2} \cdot 10}{10} = \frac{5}{10} = {}^{\backprime}5$, $\frac{2}{3} = \frac{\frac{2}{3} \cdot 100}{100} = \frac{\frac{200}{3}}{100} = \frac{66\frac{2}{3}}{100} = {}^{\backprime}66\frac{2}{3}$;

and $\frac{a}{b} = \frac{\frac{a}{b} \cdot 10^n}{10^n} = \frac{\frac{a \cdot 10^n}{b}}{10^n}$; whence the rule for reducing vulgar to decimal fractions.

Reducing $\frac{N}{D}, \frac{N_2}{D_2}, \frac{N_3}{D_3}, \ldots$ to the denominator M, we find

$\frac{\frac{MN}{D}}{M}, \frac{\frac{MN_2}{D_2}}{M}, \frac{\frac{MN_3}{D_3}}{M}, \ldots$, from which it appears that the *least* common denominator, M, must be divisible by each of the denominators, D, D_2, D_3, ... of the given fractions, or, must be the *least common multiple* of all the denominators, if we would have the resulting fractions freed from subdenominators, ∴.

Cor. 10. To add or subtract fractions, find the least common multiple of the denominators, to which, as a common denominator, reduce all the fractions, then add or subtract the numerators (5_3). (27)

This rule will frequently be superseded by the following :

Cor. 11. To free a fraction of subdenominators, multiply (28)
its terms by the least common multiple of the subdenominators (20).

Thus, $\dfrac{\frac{a}{b}+\frac{c}{d}-e}{\frac{f}{g}-\frac{h}{i}+\frac{j}{k}} = \dfrac{M \cdot \frac{a}{b}+M \cdot \frac{c}{d}-Me}{M \cdot \frac{f}{g}-M \cdot \frac{h}{i}+M \cdot \frac{j}{k}}$, where, if we would make the subdenominators b, d, g, i, k disappear, M must be divisible by each.

Scholium I. The rule for managing the signs has already been given in (6_3), but it may be well to observe that, of the three signs pertaining to a fraction—viz., the sign before the fraction and those before its terms—an even number produces plus, an odd number minus, and any two may be changed at a time. Thus,

$$+\frac{+}{+} = +(+ : +) = +(+) = +,\quad +\frac{-}{+} = +(- : +) = +(-) = -,$$

$$+\frac{-}{-} = +(- : -) = +(+) = +,\quad +\frac{+}{-} = +(+ : -) = +(-) = -;$$

$$-\frac{+}{+} = -(+ : +) = -(+) = -,\quad -\frac{-}{+} = -(- : +) = -(-) = +,$$

$$-\frac{-}{-} = -(- : -) = -(+) = -,\quad -\frac{+}{-} = -(+ : -) = -(-) = +.$$

Scholium II Whenever it may be foreseen that, by performing an operation the same factor would be introduced into the numerator and denominator, it should be suppressed.

Scholium III. Additions and subtractions will frequently be better performed in part before reducing to a common denominator.

Def. 8. An *Equation* is an algebraical expression consisting of two *members*, separated by the sign of equality [=].

Equations are of different kinds.

1°. An *identical equation* is one in which the members are the same, as $a = a$.

2°. An *equation of operation* has one of its members derived from the other—of these we have already had many examples, such as $mc - nc = (m - n)c$.

3°. An *equation of condition* expresses a determinate relation that must exist between certain quantities, not distinguished as known and unknown, as $\frac{1}{2}(a + b) = p - q$.

4°. The word *equation* more commonly indicates a relation between known and unknown quantities, such that the latter may be

derived from the former, as $x+a=b$, whence, by subtracting a from both members, we have $x=b-a$.

Equations of this kind are denominated *simple*, *quadratic*, *cubic*, &c., or of the *first*, *second*, *third*, ..., *degree*, according as the unknown quantity is of the *first*, *second*, *third*, ..., *power*.

PROPOSITION XII.

Any term may be transposed from one member of an equation to the other, by changing its sign. (29)

For, let $$A+p=B$$
be any equation, A representing the sum of all the terms in the first member, except p, which, it is to be understood, may be either $+$ or $-$, and B those of the second member; then, subtracting p from both sides (1_{II}), there results
$$A=B-p. \quad \text{Q. E. D.}$$

PROPOSITION XIII.

A Simple Equation will be solved— (30)

1°. *By separating the unknown from the known quantities by transposition ;*

2°. *By uniting the coëfficients of the unknown quantity in one ;*

3°. *By dividing by the coëfficient thus formed ;*

4°. *By clearing of subdenominators.*

Scholium. An equation may be cleared of fractions, if desired, by multiplying it by the least common multiple of all the denominators—but the solution will in general be more readily accomplished by the rule just given. The exercises marked I. at the end of the book, may be here introduced.

PROPOSITION XIV.

From n *equations of the first degree to eliminate* n— 1 *unknown quantities.*

1°. Whenever the conditions of a problem embrace two unknown quantities of the first degree, it is evident from what has

just been said that the equations expressing these conditions may be reduced to the form

$$\left.\begin{aligned} ax+by&=c,\\ a'x+b'y&=c';\end{aligned}\right\}$$

and $\therefore$

$$\left.\begin{aligned} \frac{a}{b}\cdot x+y&=\frac{c}{b},\\ \frac{a'}{b'}\cdot x+y&=\frac{c'}{b'}\end{aligned}\right\}$$

whence

$$\left(\frac{a}{b}-\frac{a'}{b'}\right)x=\frac{c}{b}-\frac{c'}{b'},$$

an equation involving but a single unknown quantity, which may therefore be solved as above. The method may obviously be extended to three or more unknown quantities, embraced in three or more independent equations—equations not convertible into each other;—and it will become manifest, by making the elimination, that the number of independent equations must equal the number of unknown quantities to be determined.

$\therefore$ *From several equations of the first degree to eliminate one or more quantities:* (31)

1°. *Make the coëfficients of the quantity to be eliminated* $+1$ *in each equation;*

2°. *Take the differences of the equations thus formed.*

Example. Given equations (a) to find x.

$$\left.\begin{aligned} 12x-8y+4z&=8 &(1)\\ 3x+y-6z&=-13 &(2)\\ x+7y+14z&=57 &(3)\end{aligned}\right\}(a)$$

$$\left.\begin{aligned} &\therefore,\ (1):4, & 3x-2y+z&=2 &(4)\\ &(2):-6, & -\tfrac{1}{2}x-\tfrac{1}{6}y+z&=2\tfrac{1}{6} &(5)\\ &(3):14, & \tfrac{1}{14}x+\tfrac{1}{2}y+z&=4\tfrac{1}{14} &(6)\end{aligned}\right\}(b)$$

$$\left.\begin{aligned} &\therefore,\ (4)-(5), & 3\tfrac{1}{2}x-1\tfrac{5}{6}y&=-\tfrac{1}{6} &(7)\\ &(4)-(6), & 2\tfrac{13}{14}x-2\tfrac{1}{2}y&=-2\tfrac{1}{14} &(8)\end{aligned}\right\}(c)$$

$$\left.\begin{aligned} &\therefore,\ (7):-1\tfrac{5}{6}, & -1\tfrac{10}{11}x+y&=\tfrac{1}{11} &(9)\\ &(8):-2\tfrac{1}{2}, & -1\tfrac{6}{35}x+y&=\tfrac{29}{35} &(10)\end{aligned}\right\}(d)$$

$$\therefore\qquad \left(-\tfrac{10}{11}+\tfrac{6}{35}\right)x=\tfrac{1}{11}-\tfrac{29}{35} \qquad (e)$$

$$\therefore\ x=\frac{\tfrac{1}{11}-\tfrac{29}{35}}{-\tfrac{10}{11}+\tfrac{6}{35}}=1.$$

Substituting 1 for x in (9), we find

$$y=\tfrac{1}{11}+1\tfrac{10}{11}=2;$$

$\therefore$ (4) $\qquad z=2+2\cdot 2-3\cdot 1=3.$

After a little practice it will be found unnecessary to write the equations (*b*) and (*d*), it being equally easy to pass at once from (*a*) to (*c*) and from (*c*) to (*e*). This method will generally produce the greatest amount of cancelation, and, therefore, be the most expeditious.

The exercises marked II. at the end of the book may be here introduced.

SECTION SECOND.

Exponents—Proportion—Variation.

PROPOSITION I.

A factor may be carried from one term to the other of a fraction by changing the sign of its exponent. (32)

As in subtracting coëfficients, we must be guided by the same rule, whatever may be their relative values, so, by a like extension in exponents, from (15) we must always have $a^m : a^n = a^{m-n}$, whether $m >$ or $< n$;

$$\therefore \quad a^{m-n} = \frac{a^m}{a^n} = \frac{a^m : a^m}{a^n : a^m} = \frac{1}{a^{n-m}} = \frac{1}{a^{-(m-n)}};$$

$$\therefore \quad \frac{Na^{+(m-n)}}{D} = \frac{N}{Da^{-(m-n)}},$$

multiplying both sides by $\frac{N}{D}$, and the proposition is proved.

Cor. 1. The rules for multiplication and division, and, consequently, for involution and evolution, are the same for minus as for plus exponents, both being supposed integral. (33)

For from $$a^{-m} = \frac{1}{a^{+m}}$$

and $$a^{-n} = \frac{1}{a^{+n}}$$

there results $a^{-m} \cdot a^{-n} = \frac{1}{a^m \cdot a^n} = \frac{1}{a^{m+n}} = a^{-(m+n)} = a^{-m-n}$,

and $a^{-m} : a^{-n} = \frac{1}{a^m} : \frac{1}{a^n} = \frac{a^n}{a^m} = a^{n-m} = a^{-m-(-n)}$,

also $[n=m]\ a^{-m} \cdot a^{-m} = a^{-m-m}$, or $(a^{-m})^2 = a^{-2m} = a^{2\cdot -m}$,

$\therefore \quad (a^{-m})^3 = (a^{-m})^2 \cdot a^{-m} = a^{2\cdot -m} \cdot a^{-m} = a^{3\cdot -m}$, and so on,

$\therefore \quad (a^{-m})^r = a^{r\cdot -m}$;

consequently $(a^{\pm m})^{-r} = \frac{1}{(a^{\pm m})^r} = \frac{1}{a^{r\cdot \pm m}} = a^{-(r\cdot \pm m)} = a^{-r\cdot \pm m}$,

$\therefore$ $(a^p)^q = a^{pq}$, whatever integers p and q may be, $+$ or $-$,

and $\therefore$ $\sqrt[q]{a^{pq}} = \sqrt[q]{(a^p)^q} = a^p = a^{(pq):q}$.

Cor. 2. It will be observed that any quantity affected by the exponent zero [0] is equal to unity [1]; since $a^m : a^m$ gives either a^0 or 1. (34)

PROPOSITION II.

The root of a power is to be taken by dividing its exponent by the index of the root. (35)

For from (14) we have $\sqrt[n]{a^{mn}} = \sqrt[n]{(a^m)^n} = a^m = a^{(mn):n}$,
which becomes $\sqrt[n]{a^r} = a^{r:n}$, putting $r = mn$; and the rule thus established must evidently be extended, for the sake of consistency, to the case in which r is not a multiple of n—just as, when we cannot execute the division of 3 by 5 we express it as a fraction, $\frac{3}{5}$, and seek appropriate rules for its management.

Cor. 1. An exponent may be reduced to a given denominator like any other quantity. (36)

As $a^{\frac{n}{m}}$ is equivalent to $\sqrt[m]{a^n}$, so $a^{\frac{rn}{rm}}$ must be regarded as $\sqrt[rm]{a^{rn}}$, in order to be consistent; and we are to show that

$$a^{\frac{n}{m}} = a^{\frac{rn}{rm}}.$$

If not, let x be such as to make

$$(a+x)^{\frac{n}{m}} = a^{\frac{rn}{rm}},$$

then, raising both sides to the mth power, observing that the mth power of the mth root is the quantity itself, we have

$$(a+x)^n = \left(a^{\frac{rn}{rm}}\right)^m, \text{ which involved again}$$

gives $$[(a+x)^n]^r = \left[\left(a^{\frac{rn}{rm}}\right)^m\right]^r,$$

or (14) $\quad (a+x)^{rn} = \left(a^{\frac{rn}{rm}}\right)^{rm} = a^{rn}$;
therefore by evolution, $a+x=a$, or $x=0$;

$\therefore \quad a^{\frac{n}{m}} = a^{\frac{rn}{rm}}$;

and the demonstration is finished, since (33) m, n, r may be either plus or minus.

Cor. 2. All real exponents, whether plus or minus, whole or fractional, are to be managed by the same rules; addition corresponding to multiplication, subtraction to division, multiplication to involution, and division to evolution. (37)

Let $a^{\frac{m}{r}}$, $a^{\frac{n}{r}}$ be any fractional powers, the exponents reduced to the same denominator, being either + or −: then will

$$a^{\frac{m}{r}} \cdot a^{\frac{n}{r}} = a^{\frac{m+n}{r}} = a^{\frac{m}{r}+\frac{n}{r}}.$$

If not, let $\quad (a+x)^{\frac{m+n}{r}} = a^{\frac{m}{r}} \cdot a^{\frac{n}{r}}$,

then $(a+x)^{m+n} = \left(a^{\frac{m}{r}} \cdot a^{\frac{n}{r}}\right)^r = \left(a^{\frac{m}{r}} \cdot a^{\frac{n}{r}}\right) \cdot \left(a^{\frac{m}{r}} \cdot a^{\frac{n}{r}}\right) \cdot \left(a^{\frac{m}{r}} \cdot a^{\frac{n}{r}}\right) \ldots [r]$,

or (4) $\quad (a+x)^{m+n} = a^{\frac{m}{r}} \cdot a^{\frac{m}{r}} \cdot a^{\frac{m}{r}} \ldots [r] \times a^{\frac{n}{r}} \cdot a^{\frac{n}{r}} \cdot a^{\frac{n}{r}} \cdot \ldots [r]$,

$$= \left(a^{\frac{m}{r}}\right)^r \cdot \left(a^{\frac{n}{r}}\right)^r = a^m \cdot a^n = a^{m+n};$$

$\therefore \quad a+x=a$, or $x=0$,

and $\quad a^{\frac{m}{r}} \cdot a^{\frac{n}{r}} = a^{\frac{m+n}{r}} = a^{\frac{m}{r}+\frac{n}{r}}$;

$\therefore \quad a^{\frac{m}{r}+\frac{n}{r}} : a^{\frac{n}{r}} = a^{\frac{m}{r}}$, or $a^{\frac{p}{r}} : a^{\frac{q}{r}} = a^{\frac{p-q}{r}} = a^{\frac{p}{r}-\frac{q}{r}}$;

also $[n=m]$, $\quad a^{\frac{m}{r}} \cdot a^{\frac{m}{r}} = a^{2 \cdot \frac{m}{r}}$, $\therefore \left(a^{\frac{m}{r}}\right)^3 = a^{2 \cdot \frac{m}{r}} \cdot a^{\frac{m}{r}} = a^{3 \cdot \frac{m}{r}}$,

and generally $\left(a^{\frac{m}{r}}\right)^t = a^{t \cdot \frac{m}{r}}$; $\therefore \left(a^{\frac{m}{r}}\right)^{-t} = \dfrac{1}{\left(a^{\frac{m}{r}}\right)^t} = \dfrac{1}{a^{t \cdot \frac{m}{r}}} = a^{-t \cdot \frac{m}{r}}$;

$\therefore \quad \sqrt[t]{a^{\frac{tm}{r}}} = \sqrt[t]{\left(a^{\frac{m}{r}}\right)^t} = a^{\frac{m}{r}} = a^{\frac{tm}{r} : t}$, t being either + or −;

$\therefore \quad \sqrt[u]{a^{\frac{v}{r}}} = a^{\frac{v}{r} : u}$, whatever real quantities u, v, and r may be.

Note. It follows that $a^{\frac{n}{m}}$ may be regarded either as the mth root of the nth power of a, $[(a^n)^{\frac{1}{m}}]$, or as the nth power of the mth root of a, $\left[\left(a^{\frac{1}{m}}\right)^n\right]$.

PROPORTION.

Def. 1. If $a = rc$, r is called the *ratio* of a to c, whatever r may be, whole or fractional, positive or negative, whether capable of being exactly expressed in rational terms or not. Thus, the ratio of 6 to 3 is 2, of 3 to 6 is $\frac{1}{2}$; of 2 to 3 is $\frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \ldots$, of $5 \cdot \sqrt{-1}$ to 5 is $\sqrt{-1}$.

Def. 2. If of four quantities, as a, c, a', c', a has the same ratio to c that a' has to c',

or, if $\qquad a = rc \quad (1^\circ)$

and $\qquad a' = rc'. \quad (2^\circ)$

then, a, c; a', c', are said to be *proportional*, or a is said to be to c as a' to c'; and we write

$$a : c :: a' : c',$$
$$\text{or } a : c = a' : c';$$

and read a is to c as a' to c'.

$\therefore$ *A proportion is an equality of ratios.* (38)

Def. 3. We call a, c, belonging to the same ratio, *homologous** terms, while a, a', as well as c, c', are denominated *analogous* terms; a, c, constitute the first, a', c' the second *couplet.*

Inverting the order of (1°), (2°),

$$\text{from } \left.\begin{aligned} a &= rc \\ a' &= rc' \end{aligned}\right\} \begin{matrix}\text{we}\\ \text{have}\end{matrix} \left\{\begin{aligned} a' &= rc' \\ a &= rc \end{aligned}\right\}, \text{ or } \left\{\begin{matrix} \text{if} & a : c :: a' : c', \\ \text{then} & a' : c' :: a : c. \end{matrix}\right\} (3^\circ)$$

Dividing (1°) and (2°) by r,

$$\text{from } \left.\begin{aligned} a &= rc \\ a' &= rc' \end{aligned}\right\} \begin{matrix}\text{we}\\ \text{have}\end{matrix} \left\{\begin{aligned} c &= \frac{1}{r}a, \\ c' &= \frac{1}{r}a' \end{aligned}\right\}, \text{ or } \left\{\begin{matrix} \text{if} & a : c :: a' : c', \\ \text{then} & c : a :: c' : a'. \end{matrix}\right\} (4^\circ)$$

Dividing (1°) by (2°), we have $a : a' = rc : rc' = c : c'$; whence, if the ratio of a to a' be r', that of c to c' will be the same;

$$\therefore \text{from } \left.\begin{aligned} a &= rc \\ a' &= rc' \end{aligned}\right\} \begin{matrix}\text{we}\\ \text{have}\end{matrix} \left\{\begin{aligned} a &= r'a', \\ c &= r'c', \end{aligned}\right\} \text{or} \left\{\begin{matrix} \text{if} & a : c :: a' : c', \\ \text{then} & a : a' :: c : c', \end{matrix}\right\} (5^\circ)$$

$$\therefore \ (3^\circ) \qquad c : c' :: a : a', \quad (6^\circ)$$
$$\text{and } (4^\circ) \qquad a' : a :: c' : c. \quad (7^\circ)$$

Whence, by comparing (1°) and (2°), (3°), (4°), (5°), (6°), (7°), there results,

---

* Ὁμός = homos = *like*, λόγος = logos = *ratio=comparison.*

PROPOSITION III.

If four quantities are proportional, they are proportional: (39)
1°. *By inversion of couplets;*
2°. *By inversion of terms;*
3°. *By inversion of both couplets and terms.*

By additions and subtractions we have,

from (1°), (2°) $\left\{\begin{array}{l} a = rc, \\ a' = rc' \end{array}\right\}, \left\{\begin{array}{c} a \pm a' = r(c \pm c'), \\ a = rc, \\ a' = rc'; \end{array}\right\}$

$$\therefore \left\{\begin{array}{l} \text{if} \quad a : c :: a' : c', \\ \therefore a \pm a' : c \pm c' :: a : c :: a' : c'; \\ \therefore a + a' : c + c' :: a - a' : c - c'; \end{array}\right\} (8°)$$

from (5°) $\left\{\begin{array}{l} a = r'a' \\ c = r'c' \end{array}\right\}, \left\{\begin{array}{c} a \pm c = r'(a' \pm c'), \\ a = r'a', \\ c = r'c'; \end{array}\right\}$

$$\therefore \left\{\begin{array}{l} \text{if} \quad a : c :: a' : c', \\ \therefore a \pm c : a' \pm c' :: a : a' :: c : c'; \\ \therefore a + c : a - c :: a' + c' : a' - c'. \end{array}\right\} (9°)$$

&c., &c., &c., $\therefore$.

PROPOSITION IV.

If four quantities are proportional: (40)
1°. *The sum or difference of the antecedents*
is to the sum or difference of the consequents,
as either antecedent
to its consequent;

2°. *The sum or difference of the terms of the first couplet*
is to the sum or difference of those of the second,
as antecedent to antecedent
or as consequent to consequent;

3°. *The sums or differences, or both sums and differences of the terms, taken in the same order, whether homologous or analogous, are proportional.*

The principle in (40), 1°, may be extended thus: if
$a : c :: a' : c' :: a'' : c'' ::$ &c., i. e., $a : c :: a' : c'$ and
$a' : c' :: a'' : c''$, &c.,

or if
$$a = rc,$$
$$a' = rc',$$
$$a'' = rc'', \text{ \&c., \&c.};$$
then $a + a' + a'' + \ldots = r(c + c' + c'' + \ldots)$;
$$\therefore \quad a + a' + a'' + \ldots : c + c' + c'' + \ldots :: a : c :: a' : c' :: a'' : c'' :: \text{\&c.}, \therefore$$

PROPOSITION V.

If any number of couplets have the same ratio: (41)
The sum of all the antecedents
is to the sum of all the consequents,
as any one antecedent
to its consequent.

We should also have
$$\pm a \pm a' \pm a'' \pm \ldots : \pm c \pm c' \pm c'' \pm \ldots :: a : c :: a' : c' :: \text{\&c.},$$
which may be enounced in words.

From (1°) $\qquad a = rc$
and (2°) reversed $\qquad rc' = a'$,
we have $\qquad arc' = rca'$,
$\therefore \quad a : c :: a' : c'$ gives $\quad ac' = a'c, \therefore$

PROPOSITION VI.

If four quantities are proportional the product of the extremes is equal to the product of the means. (42)

Cor. To change an equation into a proportion, make any two factors into which one member may be resolved the extremes and the factors of the other member of the equation the means of the proportion; or, to read an equation as a proportion, begin and end the reading in the same member. (43)

Thus $\quad 4 \cdot 3 = 2 \cdot 6,$

$$\begin{array}{ll} \text{may be read} & \left.\begin{array}{l} 4 : 2 :: 6 : 3, \\ 4 : 6 :: 2 : 3, \\ 3 : 2 :: 6 : 4, \\ 3 : 6 :: 2 : 4, \end{array}\right\} \text{or} \left\{\begin{array}{l} 2 : 4 :: 3 : 6, \\ 2 : 3 :: 4 : 6, \\ 6 : 4 :: 3 : 2, \\ 6 : 3 :: 4 : 2. \end{array}\right. \\ \text{or} & \\ \text{or} & \\ \text{or} & \end{array}$$

From (1°) and (2°) we have
$$a^m = r^m c^m,$$
and
$$a'^m = r^m c'^m;$$
$$\therefore \quad a^m : c^m :: a'^m : c'^m,$$
where m may be any quantity whatever, whole or fractional; $\therefore$

PROPOSITION VII.

If quantities are proportional, their like powers, or roots, or powers of roots, are proportional. (44)

Again (1°) and (2°) $\left\{\begin{matrix} ma = r \cdot mc, \\ na' = r \cdot nc'; \end{matrix}\right\}$ and (5°) $\left\{\begin{matrix} ma = r' \cdot ma' \\ nc = r' \cdot nc'; \end{matrix}\right\}$
where m and n may be any whatever, whole or fractional; $\therefore$

PROPOSITION VIII.

A proportion is not destroyed by taking equal multiples or submultiples of homologous or analogous terms. (45)

Thus, since	$4 : 12 :: 8 : 24,$
we have	$16 : 24 :: 32 : 48,$ [eq. mult. of anal. terms],
and	$1 : 3 :: 1 : 3,$ [eq. submult. homol. terms].
From	$a = rc,$
we have	$a\,(1 \pm m) = r \cdot (1 \pm m)\,c,$
or	$a \pm ma = r\,(c \pm mc)$; $\therefore$

PROPOSITION IX.

If any two quantities be increased or diminished by equal multiples or submultiples of those quantities, the sums or differences thus formed will be proportional to the quantities themselves. (46)

Def. 4. Four quantities are said to be *reciprocally* proportional when the ratios of the couplets are the reciprocals of each other.
Thus if $a = rc$ and $a' = \frac{1}{r}c'$; then a, c; a', c', are reciprocally proportional.
But from the same equations, we have

$$a = rc,$$
$$c' = ra'; \quad \therefore$$

Def. 5. Four quantities are *inversely* proportional when the first is to the second as the fourth to the third.

Def. 6. If $a : b :: b : c :: c : d ::$, &c., then $a, b, c, d, \ldots$, are said to be in *continued proportion*, or to constitute a *geometrical progression;* c is said to be a *third* proportional to a and b, and b is called a *mean* proportional between a and c.

Since from $a : b :: b : c$, &c., we have

$$
\begin{aligned}
ra &= b,\\
rb &= c,\\
rc &= d,\\
&\dots\ \dots\ \dots,\\
rk &= l;
\end{aligned}
$$

multiplying together the 1st and 2d, the 1st, 2d and 3d, &c., there results

$$
\begin{aligned}
r^2 a &= c,\\
r^3 a &= d,\\
&\dots\ \dots\ \dots,\\
r^{n-1} a &= l,
\end{aligned}
$$

l being the nth term ; ∴.

PROPOSITION X.

In a continued proportion we have the 1st *term to the* 3d *as the square of the* 1st *to the square of the* 2d, *or as the square of the second to the square of the* 3d, *&c. ; and the* 1st *to the* nth *as the* (n − 1)*th power of the first to the* (n − 1)*th power of the* 2d, *&c.* (47)

Thus $a : l :: a^{n-1} : b^{n-1} :: b^{n-1} : c^{n-1} :: \dots$.

Cor. The last or nth term is equal to the first multiplied by the ratio raised to a power one less than the number of the term $[n-1]$. (48)

It will be observed that the series is here supposed to be *ascending*, or that $a < b < c \dots$; if it be descending, or $a > b > c \dots$, then r will be less than unity—and in all cases

$$r = \frac{b}{a} = \frac{c}{b}, \text{ \&c.}$$

By addition we infer that

$$r(a+b+c+\dots+k) = b+c+d+\dots+l,$$

or

$$r(S-l) = S-a; \qquad [S = a+b+c+\dots+l]$$

$$S = \frac{rl-a}{r-1} = \frac{a(r^n-1)}{r-1}, \text{ or}$$

PROPOSITION XI.

$$\left.\begin{matrix}\textit{Sum of}\\ \textit{terms}\end{matrix}\right\} = \frac{(\textit{ratio})(\textit{last}) - \textit{first term}}{\textit{ratio} - \textit{one}} \tag{49}$$

$$= \frac{(\textit{first term})\,[(\textit{ratio})^n - 1]}{\textit{ratio} - \textit{one}}.$$

Note. Since from (1°) and (2°) we have

$$\frac{a}{c}=r \text{ and } \frac{a'}{c'}=r, \text{ and } \therefore \frac{a}{c}=\frac{a'}{c'},$$

we may write every proportion as a fractional equation—whereby the many and all the changes that can be rung on proportions, will be reduced to *very simple* operations on equations.

VARIATION.

Def. 7. We shall often have occasion to consider quantities, not as proportional simply, but as passing by *inappreciable* degrees through all magnitudes compatible with certain conditions. Such quantities are denominated VARIABLE—and are represented by the last letters of the alphabet, as x, y, z, while the first letters are used to indicate quantities regarded as *constant*, or such as are independent of the variables. Thus, that y varies as x, a being their constant ratio, is expressed by

$$y=ax.$$

In this equation it is to be understood that x, in passing from any one given value to another, is regarded as passing through all intermediate values while a remains unchanged; and that consequently y changes, taking new values depending upon the value of x. On this account x is called the *independent*, and y the *dependent* variable. Thus, if the rate of interest, r, be constant, and the principal, p, be also constant, we shall have a given sum pr, as the interest for one year on the given principal; then if y be the interest for the time x in years, there will result the variation

$$y=prx.$$

This manner of looking upon quantities, not so much as known and unknown, as constant and variable, is as important as it is simple.

PROPOSITION XII.

If $$y=ax,$$

$\therefore$ $$x=\frac{1}{a}y; \text{ i. e.}$$

If y *vary as* x, x *varies as* y. (50.)

PROPOSITION XIII.

If $y = ax$,

$\therefore \quad y \pm x = ax \pm x = (a \pm 1)\, x = (a \pm 1) \cdot \frac{1}{a} \cdot y$; i. e.,

If y *vary as* x, x ± y *varies as* x *or* y. (51)

PROPOSITION XIV.

If $y = ax$,

$\therefore \quad y^m = a^m x^m$; i. e.,

If y *vary as* x, y^m *varies as* x^m. (52)

PROPOSITION XV.

If $y = ax$,

$\therefore \quad my = a \cdot mx$; i. e.,

If y *vary as* x, my *varies as* mx. (53)

PROPOSITION XVI.

$y = amx$, it follows that

If y *vary as* x, y *also varies as* mx.

PROPOSITION XVII.

If $y = az$,

and $z = bx$;

$\therefore \quad y = abx$; i. e.,

If y *vary as* z, *and* z *vary as* x; *then* y *varies as* x. (54)

PROPOSITION XVIII.

If $y = az$,

and $x = bz$;

$\therefore \quad y \pm x = (a \pm b)\, z$; i. e.,

If y *and* x *vary as* z, y ± x *varies as* z. (55)

It will be observed that the above forms embrace, very briefly and simply, not only essentially the whole doctrine of proportion, but a vastly wider field, by reason of the unlimited number of values of which y and x are susceptible.

SECTION THIRD.

Analysis of Equations.

The following is a principle of the first importance in analytical investigations:

PROPOSITION I.

Certain equations are so constituted that they necessarily resolve into, and are consequently equivalent to, several independent equations. (56)

We do not propose, in the present article, to enter into a full developement of the boundless resources which this principle affords, but simply to illustrate it by examples of such particular cases as will be serviceable to us as we proceed.

Required two numbers such that if they be diminished severally by a, b, and the remainders squared, the sum of these squares shall be equal to zero.

Denoting the numbers by x, y, we have

$$(x-a)^2+(y-b)^2=0;$$

and it follows from (6_3) that $(x-a)^2$, $(y-b)^2$, must both be $+$, whether $x \gtrless a$,* $y \gtrless b$; but it is obvious that, since neither of the terms $(x-a)^2$, $(y-b)^2$ can be minus, neither can be greater than 0; for if either term, $(x-a)^2$ for instance, have an additive value, the other $(y-b)^2$ must possess the same value and be subtractive, in order to satisfy the equation, or that their sum may $=0$; whence it is necessary that

$$(x-a)^2=0, \text{ and } (y-b)^2=0;$$

$$\therefore \quad x-a=0, \text{ and } y-b=0,$$

or $$x=a, \text{ and } y=b.$$

As a second example, what numbers are those from which if a and b be severally subtracted, the product of the remainders will be $=0$?

Representing the numbers by x, y, there results

$$(x-a)(y-b)=0;$$

$\therefore$ dividing by $y-b$, $\quad x-a=0$, or $x=a$,

and dividing by $x-a$, $\quad y-b=0$, or $y=b$.

---

* $x \gtrless a$, x greater or less than a.

Every equation of the *second degree*, or, embracing no other unknown quantities than x^2 and x, may, by transposing, uniting terms, and dividing by the coëfficient of x^2, be readily reduced to the form

$$+1 \cdot x^2 + 2ax = b\,; \tag{57}$$

understanding that a, b, may be either $+$ or $-$. In order to find x, we observe that the first member of the equation will become a binomial square (8) by the addition of a^2;

$\therefore$ $\quad x^2 + 2xa + a^2 = b + a^2,$

$\therefore$, (8), (6_4), $\quad x + a = \pm\sqrt{(b + a^2)},$

and $\quad x = -a \pm \sqrt{(b + a^2)}\,;\ \therefore,$ (58)

PROPOSITION II.

To solve a QUADRATIC EQUATION:

1°. *Reduce it to the form of* (57);

2°. *Add the square of half the coëfficient of* x *to both sides;*

3°. *Take the square root of the members, prefixing the double sign* [$\pm$] *to the second;*

4°. *Transpose.*

It will be observed that (57) resolves into two independent equations (58),

$$x = -a + \sqrt{(b + a^2)}, \text{ and } x = -a - \sqrt{(b + a^2)},$$

thus illustrating (56).

Adding these values of x, we have

$$[-a + \sqrt{(b + a^2)}] + [-a - \sqrt{(b + a^2)}] = -2a,$$

and their product is (?)

$$[-a + \sqrt{(b + a^2)}][-a - \sqrt{(b + a^2)}] = [-a]^2 - [\sqrt{(b + a^2)}]^2$$
$$= a^2 - (b + a^2) = -b.$$

Cor. In a quadratic of the form (57) the sum of the values (59) of x, is equal to the coëfficient of x^1 taken with the contrary sign, and their product to the second member also taken with the contrary sign.

Queries. What will (57) and (58) become, when a is changed into $-a$? b into $-b$? When b is minus, what must be its value compared with a^2 in order that the value of x in (58) be impossible? [See (6_5.)] Will change of value or sign in a ever render x imaginary? Why not?

Scholium I. We naturally inquire if the *Cubic Equation* can be

resolved into *three* independent equations. Every equation of the third degree reduces to the form

$$x^3+ax^2+bx+c=0. \qquad (a)$$

As we must suppose x to have some value—one at least—let r be that value; then must r satisfy the equation,

or $$r^3+ar^2+br+c=0;$$

$$\therefore \quad c=-r^3-ar^2-br,$$

which value of c substituted above, gives

$$\begin{aligned}x^3+ax^2+bx\\ -r^3-ar^2-br=0,\end{aligned}$$

or $$(x^3-r^3)+a(x^2-r^2)+b(x-r)=0; \qquad (b)$$

$$\therefore \quad (x^2+xr+r^2)+a\,(x+r)+b=0,$$

dividing by $x-r$, see *examples* under (16);

$$\therefore \quad x^2+(a+r)\,x+r^2+ar+b=0, \qquad (c)$$

whence [(57), (58)] (a) is resolvable into three equations, giving three values for x.

Comparing equations (a), (b), (c), and observing that (b) is the same as (a), we learn that, if r be a root of the cubic equation (a), that the equation is divisible by $x-r$, giving a quadratic (c).

The student is requested to prove that the equation

$$x^4+px^3+qx^2+rx+s=0,$$

is resolvable into the factors

$$x-a,\ x-b,\ x-c,\ x-d,$$

or that $(x-a)\,(x-b)\,(x-c)\,(x-d)=x^4+px^3+qx^2+rx+s=0,$ a, b, c, d, being values of x, or $x=a$, $x=b$, $x=c$, $x=d$.

Scholium II. Many Biquadratic Equations may be reduced as Quadratics; *e. g.*

1°. When reducible to the form

$$(x^2+Ax+B)\,(x^2+A'x+B')=0, \qquad (60)$$

where the conditions are

$$x^4+(A+A')x^3+(AA'+B+B')x^2+(AB'+A'B)x+BB'=0,$$

or $s=BB'$, $q=AA'+B+B'$, $p=A+A'$, $r=AB'+A'B$.

2°. When reducible to the form

$$(x^2+Ax)^2+B(x^2+Ax)^1=C, \qquad (61)$$

where the conditions are

$$x^4+2Ax^3+(A^2+B)x^2+ABx=C,$$

or $p=2A$, $r=AB$, $q=A^2+B$.

3°. When reducible to the form (61$_2$)

$$x^2+\frac{a^2}{x^2}+x+\frac{a}{x}=b,$$

for $$2 \cdot x \cdot \frac{a}{x} = 2a;$$

$$\therefore \quad \left(x+\frac{a}{x}\right)^2 + \left(x+\frac{a}{x}\right)^1 = b + 2a.$$

Example. Given $x^4 + 4x^3 + 6x^2 + 4x = 15$, to find x.

We have $A = \frac{p}{2} = \frac{4}{2} = 2$, $B = \frac{r}{A} = \frac{4}{2} = 2$, and $\left\{\begin{matrix} A^2 + B = q, \\ \text{or } 2^2 + 2 = 6; \end{matrix}\right\}$

$$\therefore \quad (x^2 + 2x)^2 + 2(x^2 + 2x)^1 = 15,$$
$$\therefore \quad (x^2 + 2x)^2 + 2(x^2 + 2x)^1 + 1 = 16,$$
$$\therefore \quad x^2 + 2x + 1 = \pm 4,$$
$$\therefore \quad x + 1 = \pm\sqrt{\pm 4} = \pm 2 \text{ or } = \pm\sqrt{-4},$$
$$\therefore \quad x = +1,\ -3,\ -1 + \sqrt{-4}, \text{ or } -1 - \sqrt{-4}.$$

How many values has x? What values are imaginary? Verify by substituting these values in the first equation.

Scholium III. When a problem embraces several unknown quantities, it may be solved by representing these quantities in different ways; but the elegance and facility of the solution will frequently depend upon the notation which we employ. The following rule, taken from the Cambridge Mathematics, (application of Algebra to Geometry,) will be serviceable.

"*If among the quantities which would, when taken each* (62)
for the unknown quantity, serve to determine all the other quantities, there are two which would in the same way answer this purpose, and it could be foreseen that each would lead to the same equation (the signs + and − excepted); then we ought to employ neither of these, but take for the unknown quantity one which depends equally upon both; that is, their half sum, or their half difference, or a mean proportional between them, or, &c."

Thus, suppose it were required to find two numbers such that their sum should be 4 and the sum of their 4th powers 82. Instead of taking x and y for the numbers, we may make them both depend equally upon x by putting

$$2 + x = \text{greater No.},$$
$$2 - x = \text{less No.};$$

since the sum would be $4 = \text{greater No.} + \text{less No.}$,

$$\therefore \quad (2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4,$$
$$\text{and} \quad (2-x)^4 = 16 - 32x + 24x^2 - 8x^3 + x^4;$$
$$\therefore \quad 2x^4 + 2 \cdot 24x^2 + 2 \cdot 16 = (2+x)^4 + (2-x)^4 = 82,$$
$$\text{or} \quad x^4 + 24x^2 = 41 - 16 = 25;$$

$$\therefore \quad x^4 + 24x^2 + 12^2 = 25 + 144 = 169,$$

$$\therefore \quad x^2 + 12 = \pm 13,$$

$$\therefore \quad x = \pm(\pm 13 - 12)^{\frac{1}{2}} = \pm 1, \text{ or } = \pm(-25)^{\frac{1}{2}};$$

$$\therefore \quad \text{greater No.} = 2 + x = 3,\ 1,\ 2 + (-25)^{\frac{1}{2}}, \text{ or } 2 - (-25)^{\frac{1}{2}},$$

and $\quad \text{less No.} = 2 - x = 1,\ 3,\ 2 - (-25)^{\frac{1}{2}}, \text{ or } 2 + (-25)^{\frac{1}{2}},$
where it must be observed that if 3 be taken for the value of the greater number, 1 must be taken for that of the less, and so on for the three remaining corresponding values.

As a second example, suppose the equations

$$x + y + (x^2 + a^2)^{\frac{1}{2}} + (y^2 + a^2)^{\frac{1}{2}} = b,$$

and $$\frac{x}{a} = \frac{a}{y},$$

given to find x any y. We shall make x and y depend equally upon the same unknown quantity by putting

$$\frac{x}{a} = \frac{a}{y} = z;$$

whence $$x = az, y = \frac{a}{z},$$

which values substituted in the first equation, give

$$az + \frac{a}{z} + (a^2 z^2 + a^2)^{\frac{1}{2}} + \left(\frac{a^2}{z^2} + a^2\right)^{\frac{1}{2}} = b,$$

or $$z + \frac{1}{z} + (z^2 + 1)^{\frac{1}{2}} + \left(\frac{1}{z^2} + 1\right)^{\frac{1}{2}} = \frac{b}{a} = n,$$

dividing by a and putting $\frac{b}{a} = n$, to avoid fractions;

$$\therefore \quad \left(1 + \frac{1}{z}\right)(z^2 + 1)^{\frac{1}{2}} = n - z - \frac{1}{z},$$

$$\therefore \quad z^2 + 2z + 1 + 1 + \frac{2}{z} + \frac{1}{z^2} = n^2 - 2nz - \frac{2n}{z} + z^2$$
$$+ 1 + 1 + \frac{1}{z^2},$$

$$\therefore \quad (2n + 2)z + \frac{2n + 2}{z} = n^2,$$

$$\therefore \quad z^2 - \frac{n^2}{2(n + 1)} \cdot z = -1,$$

or $$z^2 - 2mz = -1, \text{ putting } m = \frac{n^2}{4(n + 1)} = \frac{(\frac{1}{2}n)^2}{n + 1};$$

$$\therefore \quad z^2 - 2mz + m^2 = m^2 - 1 = (m+1)(m-1),$$

$$\therefore \quad z = m \pm [(m+1)(m-1)]^{\frac{1}{2}}.$$

Scholium IV. From notions derived from the practice of arithmetic, the student, on first being introduced to problems capable of double solution, is frequently inclined to question the propriety of the second value of x; and, consequently, to reject or neglect the second or *minus* sign, in taking the square root of a quadratic equation. Such neglect must *never* be allowed; for the double sign ($\pm$) is *necessary*—and, so far from being any thing like an excrescence or deformity in algebraical language, is a symbol of the greatest utility, as will appear hereafter when we come to apply the foregoing principles to Geometry. The following problem, illustrative of the use of the double sign, will suffice for the present.

Required the point in the line joining the centres of the earth and moon where their attractions are equal.

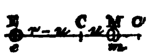

Fig. 1.

Let E be the earth, M the moon and C the point of equal attraction, and let the quantity of matter in the earth be represented by e and that in the moon by m, the radius of the moon's orbit, or distance from the earth, by r, the distance C M by u and consequently, C E by $r - u$.

Now it is a principle of physics that the attraction of a sphere is proportional to the quantity of matter directly and to the square of the distance to its centre inversely, or

$$\text{to } \frac{\text{qt. of mat.}}{(\text{dist.})^2}; \text{ whence}$$

$$\text{Earth's attraction at C will } = \frac{e}{(r-u)^2},$$

$$\text{Moon's attraction at C will } = \frac{m}{u^2};$$

$$\therefore \quad \frac{e}{(r-u)^2} = \frac{m}{u^2}$$

$$\therefore \quad \frac{\sqrt{e}}{r-u} = \pm \frac{\sqrt{m}}{u},$$

$$\therefore \quad u\sqrt{e} = \pm r\sqrt{m} \mp u\sqrt{m},$$

$$\therefore \quad (\sqrt{e} \pm \sqrt{m})u = \pm r\sqrt{m},$$

$$\therefore \quad u = \frac{\pm r\sqrt{m}}{\sqrt{e} \pm \sqrt{m}} = \frac{r}{1 \pm \sqrt{\frac{e}{m}}}.$$

But the mass of the moon, inferred from her action in raising the tides, is $\frac{1}{75}$ of that of the earth ;*

$$\therefore \qquad \frac{e}{m} = 75, \text{ and } \sqrt{\frac{e}{m}} = \sqrt{75} = 8{\cdot}66\,;$$

$$\therefore \qquad u = \frac{r}{1 \pm 8{\cdot}66}, \text{ or } = \frac{1}{9{\cdot}66} \cdot r, \text{ or } = -\frac{1}{7{\cdot}66} \cdot r,$$

or the distance of the point of equal attraction from the moon, is nearly $+\frac{3}{29}$ of the moon's radius, or $-\frac{3}{23}$ of moon's radius, where it remains to interpret the $-$ sign in the second answer. If we subtract from u (that is, from C lay off a line in the direction of M) a quantity less than u, the point of equal attraction will approach M, and the more, the greater the line subtracted, so that, when the line subtracted is $=$ CM, u will $= 0$, and when $>$ CM, as CC′, u will become $-$, being $= \text{MC} - \text{CC}' = \text{MC} - (\text{CM} + \text{MC}') = -\text{MC}'$. Therefore if $u = +\frac{3}{29} \cdot r$, gives the point C on the left or this side, $-\frac{3}{23} \cdot r$ corresponds to the point C′ on the right or beyond the moon. Indeed, had we supposed the point of equal attraction at C′ instead of at C, there would have resulted

$$\frac{e}{(r+u)^2} = \frac{m}{u^2} \text{ whence } u = \frac{r}{-1 \pm \sqrt{\frac{e}{m}}},$$

or
$$u = \frac{r}{-1 + 8{\cdot}66} = +\frac{1}{7{\cdot}66} \cdot r, \text{ instead of } u = -\frac{1}{7{\cdot}66} \cdot r.$$

The two values of u are therefore perfectly applicable, and, in fact, the problem would not be completely solved without them. The second value has revealed a truth, viz.: that there is a second point of equal attraction beyond the moon—which, though not contemplated in the statement of the problem, is yet a direct consequence of the equation drawn from it; and which, when once discovered by the aid of algebraical symbols, recommends itself to the understanding without them, since the attraction of the earth is greater than that of the moon. Suppose, for a simple illustration, the masses of E and M were as 9 and 4, and their distances to C′ as 3 and 2; then their attractions at C′ would be as $\frac{9}{3^2}$ and $\frac{4}{2^2}$, and consequently equal.

As a last example in which a single equation is equivalent to two independent equations, we give the following important theorem:

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* Poisson, Traité de Mécanique, p. 258.

PROPOSITION III.

Whenever a problem gives an equality between constant and variable terms, the variables being capable of indefinite diminution so as to become less than any assignable quantity; two independent equations will be formed, one between the constants, the other between the variables. (63)

Thus, let $$A+X=B+Y$$
represent any equation in which A, B, are the sums of the constant, those of the variable terms in each member, X and Y, being capable of continually decreasing and becoming less than any assignable quantity, that is less than any assignable part of A or B. Then will $$A=B, \text{ and } X=Y;$$
for, if possible, let A be greater or less than B; first suppose $A > B$ by some quantity D, or $$A=B+D;$$
$$\therefore \quad X=Y-D,$$
or $$Y=D+X,$$
whence it appears that Y can never be less than D, even should X become actually $=0$; therefore Y is greater than, or at least as great as, an assignable quantity D, which is contrary to the hypothesis, and this contradiction holds as long as D has any value, or as long as A is $> B$. Again suppose
$$A<B, \text{ or } A+D=B;$$
$$\therefore \quad X-D=Y,$$
or $$X=D+Y;$$
whence X is always greater than, or at least as great as D, and this is also contrary to the hypothesis, according to which X, as well as Y, is to be capable of indefinite diminution. Therefore it is *absurd* to suppose A either greater or less than B, or otherwise than equal to B;
$$\therefore \quad A=B, \text{ and consequently } X=Y,$$
which was to be proved.

The method of proof here employed is called that of "*reductio ad absurdum*," leading to an absurdity; where, instead of showing directly that the proposition must be true, we show that it can not be otherwise than true.

As an example, let it be required to find the sum of the repeating decimal fraction
$$\cdot 333 \text{ \&c.} = \cdot 3 + \cdot 03 + \cdot 003 + \ldots \ [\textit{ad infinitum}].$$

Let $A =$ sum of all the terms,
$X =$ sum of term after the nth;
then $A - X$ will $=$ sum of n first terms,
or $A - X = {}^{\cdot}3 + {}^{\cdot}03 + {}^{\cdot}003 + \ldots$ [n terms],

$$\therefore (49) \quad A - X = \frac{{}^{\cdot}3[({}^{\cdot}1)^n - 1]}{{}^{\cdot}1 - 1} = \frac{{}^{\cdot}3[1 - ({}^{\cdot}1)^n]}{1 - {}^{\cdot}1} = \frac{{}^{\cdot}3}{{}^{\cdot}9} - \frac{({}^{\cdot}1)^n}{{}^{\cdot}9}.$$

But $({}^{\cdot}1)^n$ may be made less than any assignable quantity by taking n sufficiently great, which may be done since the number of terms is infinite. Thus, if we take 1, 2, 3, &c., terms, or make successively

$$n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad \ldots,$$

we have $({}^{\cdot}1)^n = {}^{\cdot}1, {}^{\cdot}01, {}^{\cdot}001, {}^{\cdot}0001, {}^{\cdot}00001, \ldots$;

$\therefore$, $\dfrac{({}^{\cdot}1)^n}{{}^{\cdot}9}$ and X becoming $<$ any assignable quantity, we have (63)

$$A = \frac{{}^{\cdot}3}{{}^{\cdot}9} = \frac{3}{9}.$$

So for the circulate ${}^{\cdot}23$, 23, 23, &c.,

we have $A - X = {}^{\cdot}23 + {}^{\cdot}0023 + {}^{\cdot}000023 + \ldots$ [n],

or $$A - X = \frac{{}^{\cdot}23}{{}^{\cdot}99} - \frac{({}^{\cdot}01)^n}{{}^{\cdot}99};$$

$$\therefore \quad A = \frac{{}^{\cdot}23}{{}^{\cdot}99} = \frac{23}{99}.$$

And generally, if $C =$ any circulate containing c digits, we shall have

$$A - X = C({}^{\cdot}1)^c + C({}^{\cdot}1)^{2c} + C({}^{\cdot}1)^{3c} + \ldots [n];$$

$$\therefore \quad A = \frac{C({}^{\cdot}1)^c}{1 - ({}^{\cdot}1)^c} = \frac{C}{(10)^c - 1} = \frac{C}{999 \ldots [c]}.$$ Why? Rule?

We observe that a quantity may increase constantly by an infinite number of additions, and yet never exceed a determinate finite quantity or *limit;* as ${}^{\cdot}333$, &c. constantly approaches to $\frac{3}{9}$ which it can never surpass. This gives us our first notion of *limits.*

Def. "When a variable magnitude $A - X$ can be made to approach another A which is fixed in such a way as to render their difference X lessthan any given magnitude, without, however, their being able ever to become rigorously equal, the second A is called a *limit* of the first $A - X$."* (64)

A direct consequence of (63) and further illustrating (56), is the following important principle:

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* Francœur, Mathematiques pures, p. 166.

Cor. If an equation exist between terms affected by whole (65) additive powers of a variable quantity, x, capable of indefinite diminution, then the constant coëfficients of the like powers of x, will be respectively equal to each other.

Thus, if $A + A_1 \cdot x^1 + A_2 \cdot x^2 + A_3 \cdot x^3 + \ldots$

$$= a + a_1 \cdot x^1 + a_2 \cdot x^2 + a_3 \cdot x^3 + \ldots,$$

where A, A_1,* A_2, A_3, ..., a, a_1, a_2, a_3, ..., are constant and independent of x which is variable, then

(63)† $A = a$, and $A_1 \cdot x + A_2 \cdot x^2 + \ldots = a_1 \cdot x + a_2 \cdot x^2 + \ldots,$

or $A_1 + A_2 \cdot x + A_3 \cdot x^2 + \ldots = a_1 + a_2 \cdot x + a_3 \cdot x^2 + \ldots;$

and $\therefore$ (63) $A_1 = a_1$, &c., &c., &c.; which was to be proved.

If, in a problem, a, a_1, a_2, ..., can by any means be found, A, A_1, A_2, ..., will become known. Applications will be given further on.

EXERCISES.

I. *In Simple Equations.*

1°. Given $x + 3 = 5$, to find x. Subtracting 3 from both sides, we have

$$x = 5 - 3 = 2, \text{ the } Ans.$$

2°. Given $x - 5 = 7$, to find x.

Adding 5, $x = 7 + 5 = 12$, *Ans.*

3°. Given $2x = 15 + x$;

$\therefore$ $2x - x = 15$,

or, found, $x = 15$.

4°. Given $2x - 25 = 3x - 75$;

$\therefore$ $75 - 25 = 3x - 2x$,

or $3x - 2x = 75 - 25$,

or $x = 50$, found.

5°. Given $x - a = b$,

found $x =$ what?

6°. Given $4x + m = 3x + n$,

found $x =$ what?

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* Read A sub-one, A sub-two, &c., or simply A first, A second, &c. The advantage of this notation is that it points out the power of x to which the coefficient belongs; thus, A_n would belong to x^n.

† For, denoting by A_q the greatest of the coëfficients A_1, A_2, A_3, ..., we have

$$A_1 \cdot x + A_2 \cdot x^2 + A_3 \cdot x^3 + \ldots < A_q \cdot x + A_q \cdot x^2 + A_q \cdot x^3 + \ldots$$
$$= A_q \cdot x\,(1 + x + x^2 + \ldots + x^n)$$

(49) $$= A_q \cdot x \cdot \frac{1 - x \cdot x^n}{1 - x},$$

which diminishes without limit as x approaches zero—and the same may be affirmed of

$$a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3 + \ldots.$$

7°. Given $(m+1)x-p=mx+q$,
found $x=p+q$.

8°. Given $\frac{9}{7}x-\frac{3}{4}=\frac{2}{7}x+\frac{5}{4}$,
found $x=?$

9°. Given $2x=8$,
found $x=4$.

10°. Given $3x=9$,
found $x=3$.

11°. Given $4x=3$,
found $x=\frac{3}{4}$.

12°. Given $mx=n$,
found $x=\frac{n}{m}$.

13. Given $ax=ab$,
found $x=b$.

14. Given $ax=ab-ac$,
found $x=?$

15°. Given $\frac{m}{n}x=\frac{p}{q}$.

found $x=\frac{p}{q}:\frac{m}{n}=\frac{p}{q}\cdot\frac{n}{m}=\frac{pn}{qm}$.

16°. Given $\frac{x}{r}=\frac{s}{t}$, or $\frac{1}{r}\cdot x=\frac{s}{t}$,

found $x=\frac{s}{t}:\frac{1}{r}=\frac{s}{t}\cdot r=\frac{sr}{t}$.

17°. Given $ax+bx=c$,
$\therefore\quad (a+b)x=c$,
$\therefore\quad x=\frac{c}{a+b}$.

18°. $ax+b^2=a^2+bx$,
$\therefore\quad x=a+b$.

19°. $\frac{x}{m}=a+b$,
$\therefore\quad x=m(a+b)=ma+mb$.

20°. $\frac{x}{m+n}=a+b+c$,
$$\therefore\quad x=(m+n)(a+b+c)$$
$$=ma+mb+mc$$
$$+na+nb+nc.$$

21°. $\frac{x}{a+b-c}=a+b+c$,
$\therefore\quad x=(a+b)^2-c^2$.

22°. $\frac{px}{a+b-c+d}=a+b+c-d$,
$$\therefore\quad x=\frac{(a+b)^2-(c-d)^2}{p}$$

23°. $$\frac{p^2+2qr-(q^2+r^2)}{x}=p+q-r,$$
$$\therefore\quad p^2+2qr-(q^2+r^2)=(p+q-r)x,$$
$$\therefore\quad x=p-q+r.$$

24°. $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}$, $\therefore\ x=9$.

25°. $2x+7+\frac{3}{2}x=6x-23$, $\therefore\ x=12$.

26°. $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3x}{4}-5\frac{3}{8}$, $\therefore\ x=139\frac{1}{2}$.

II. *In Elimination.*

1°. Given $\left\{\begin{matrix}2x+3y=14\\5x+4y=21\end{matrix}\right\}$ $\therefore$ $\left\{\begin{matrix}\frac{2}{3}x+y=4\frac{2}{3}\\\frac{5}{4}x+y=5\frac{1}{4}\end{matrix}\right\}$ $\therefore$ $(\frac{5}{4}-\frac{2}{3})x=1\frac{1}{4}-\frac{2}{3}$,

$$\therefore \qquad x=\frac{1\frac{1}{4}-\frac{2}{3}}{\frac{5}{4}-\frac{2}{3}}=\frac{15-8}{15-8}=1,\text{ and } y=4.$$

2°. $x+2y+3z=14,$
$4x+5y+6z=32,$
$6x+5y+4z=28;$
$\therefore$ $x=?$ $y=?$ $z=?$

3°. $\frac{x}{2}+\frac{y}{3}=4,$
$\frac{x}{4}+\frac{y}{6}=2;$
$\therefore$ $x=?$ $y=?$

4°. $ax+by=c,$
$a'x+b'y=c';$
$\therefore$ $x=?$ $y=?$

5°. $ax-by=bc,$
$px-qy=qr;$
$\therefore$ $x=?$ $y=?$

III. *In Quadratics.*

1°. Given $\qquad x^2+4x=45$, to find x.
Adding 4, $\qquad x^2+4x+4=49,$
or, $\qquad (x+2)^2=49,$
$\therefore \qquad x+2=\pm 7,$
$\therefore \qquad x=-2\pm 7=5$, or $=-9$.

2°. $\qquad 3x^2+6=x^2+24,\ \therefore\ x=9.$

3°. $\qquad ax^2-b=cx^2+d,\ \therefore\ x=?$

4°. $\qquad 7-x^2-8x^2=6x^2+3-21,\ \therefore\ x=?$

5°. $\qquad \frac{x^2}{a}-\frac{x}{b}=c,\ \therefore\ x=?$

6°. $\qquad (x-a)^2+2n(x-a)=b,\ \therefore\ x=a-n\pm(b+n^2)^{\frac{1}{2}}$

7°. $x^{2n}+rx^n=a$, or $y^2+2my=a$, putting $y=x^n$, $2m=r$;

$$\therefore\ y=-m\pm(a+m^2)^{\frac{1}{2}},\ \therefore\ x=y^{\frac{1}{n}}=\left[-m\pm\left(a+\frac{x^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{n}}.$$

8°. $\qquad x^6-6x^3=16,\ \therefore\ x=?$

10°. $\qquad z^2-2az=2ab+b^2,\ \therefore\ x=?$

11°. $\qquad ru^2-2rmu=2rnu+p^2r-2rpm-2nrp,\ \therefore\ x=?$

12°. $\qquad x^4+36+x^3+6x-18x^2=0,\ \therefore\ x=?$

IV. *Eliminations producing Quadratics.*

1°. $\left\{\begin{array}{l} xy = 750 \\ \dfrac{x}{y} = 3\frac{1}{3}. \end{array}\right\}$ 2°. $\left\{\begin{array}{l} xy = a \\ \dfrac{x}{y} = b. \end{array}\right\}$ 3°. $\left\{\begin{array}{l} x^2 + y^2 = 13001 \\ x^2 - y^2 = 1449. \end{array}\right\}$

4°. $\left\{\begin{array}{l} x^2 + y^2 = a \\ x^2 - y^2 = b. \end{array}\right\}$ 5°. $\left\{\begin{array}{l} \dfrac{x}{y} = \frac{3}{4} \\ x^2 + y^2 = 324900. \end{array}\right\}$ 6°. $\left\{\begin{array}{l} \dfrac{x}{y} = \dfrac{m}{n} \\ x^2+y^2=b. \end{array}\right\}$

7°. $\left\{\begin{array}{l} xy = a \\ xz = b \\ y^2 + z^2 = c. \end{array}\right\}$ 8°. $\left\{\begin{array}{l} \dfrac{xy}{z} = a \\ \dfrac{xz}{y} = b \\ \dfrac{yz}{x} = c. \end{array}\right\}$ 9°. $\left\{\begin{array}{l} xy = a \\ yz = b \\ xz = c. \end{array}\right\}$

10°. $\left\{\begin{array}{l} x + y = 500 \\ \dfrac{450-x}{450-y} = \dfrac{2x}{5y}. \end{array}\right\}$ 11°. $\left\{\begin{array}{l} x + y = 2000 \\ \dfrac{1710-x}{1040-y} = \dfrac{17x}{12y}. \end{array}\right\}$ 12°. $\left\{\begin{array}{l} x+y=41 \\ x^2+y^2=901. \end{array}\right\}$

13°. $\left\{\begin{array}{l} x + y = a \\ x^2+y^2=b. \end{array}\right\}$ 14°. $\left\{\begin{array}{l} xy = 255 \\ x^2+y^2=514. \end{array}\right\}$ 15°. $\left\{\begin{array}{l} \dfrac{x}{y}=\dfrac{m}{n} \\ (a-x)\,(b-y)=p \end{array}\right\}$

16°. $\left\{\begin{array}{l} x : y = m : n \\ (a-x)^2 + (b-y)^2=s. \end{array}\right\}$ 17°. $\left\{\begin{array}{l} x^2 - y^2 + x - y = 150 \\ x^2 + y^2 + x + y = 330. \end{array}\right\}$

18°. $\left\{\begin{array}{l} x + y = xy \\ x^2 - y^2 = xy. \end{array}\right\}$ 19°. $\left\{\begin{array}{l} x + xy + xy^2 = 126 \\ x \cdot xy \cdot xy^2 = 13824. \end{array}\right\}$

20°. $\left\{\begin{array}{l} (x - y)\,(x^2 - y^2) = 160 \\ (x + y)\,(x^2 + y^2) = 580. \end{array}\right\}$ 21°. $\left\{\begin{array}{l} x + y + xy = 34 \\ x^2+y^2-(x+y)=42. \end{array}\right\}$

22°. $\left\{\begin{array}{l} x + y + xy = a \\ x^2 + y^2 - (x + y) = b. \end{array}\right\}$ 23°. $\left\{\begin{array}{l} x + y = a \\ x^5 + y^5 = b. \end{array}\right\}$

24°. $\left\{\begin{array}{l} x + y = a \\ xy\,(x^2 + y^2) = b. \end{array}\right\}$ 25°. $\left\{\begin{array}{l} x^2 + y^2 + x + y = a \\ m\,(x^2+y^2) + nxy = b. \end{array}\right\}$

Axiome = unconditional theorem
Postulate = " " problem

BOOK SECOND.

PLANE GEOMETRY DEPENDING ON THE RIGHT LINE.

SECTION FIRST.

Comparison of Angles.

Def. 1. The doctrines of extension constitute the science of Geometry.

Def. 2. Solids have three dimensions, length, breadth, and thickness.

Def. 3. The boundaries of a solid are surfaces, the perimeter of a surface are lines, and the extremities of a line are points.

PROPOSITION I. [PRIMARY NOTION].

A Straight Line is such that it does not change its direction at any point in its whole extent. (66)

This truth is not produced as a theorem, for it is incapable of demonstration; not as a problem, for there is nothing to be done; neither as a corollary, for it is the consequence of nothing; nor as an axiom, for it is hardly of the unconditional and absolute character of that enounced in the words, "the whole is equal to the sum of all its parts;" and it is not a definition, for we gain no new idea by the mere terms of the proposition: we only recognize by and in them one of those *primary notions* which we possess anterior to all instruction, and which, as they are necessary *to*, lie at the foundation *of*, every logical deduction. It would doubtless be out of place to enter here into any investigation in regard to the origin of our ideas; but I think it will be apparent, that the notion of *continuity* is coöriginal with that of personal identity, and therefore, antecedent to argumentation; and continuity measured out on

the one hand, in the lapse of events, as the periodic return of day and night, the revolutions of the celestial sphere, or the index of the chronometer, becomes time, and, on the other, attached to form by a personal passage over the surfaces of bodies, it becomes space; and, considered in regard to space and restricted by the notion of perfect sameness, continuity gives us the idea of *direction*, or that of the *straight line*.

Corollary 1. Two straight lines cannot intersect in more (67)
points than one; for having crossed once, it is obvious from (66), that, in order to a second intersection, one of the lines, at least, must change its direction.

Cor. 2. Straight lines coinciding in two points, coincide (68)
throughout, otherwise two straight lines would intersect in more points than one, which is contradictory to (67).

Cor. 3. Straight lines coinciding in part, coincide through- (69)
out, and form one and the same straight line (68). (Why?)

Cor. 4. Two straight lines cannot include a space (68). (70)
(Why?)

Def. 4. A Plane is a surface with which a straight line may be made to coincide in any direction.

Def. 5. Two straight lines are said to be parallel when, situated in the same plane, they do not meet how far soever they may be produced.

Cor. 5. Through the same point, only one straight line can (71)
be drawn parallel to a given line;* for the directions of all the lines save one, drawn through the given point, will evidently (66) be such as to cause them to meet the given line if sufficiently produced.

Scholium. The last corollary, though commonly given as an axiom, has been thought not sufficiently evident of itself.

It is not self evident, doubtless, if regarded as a consequence of any mere *definition* that can be given of a straight line, but *necessarily* follows from that idea of a straight line, viz., *continuity in sameness of direction*, which we possess anterior to all definition. As such, it is, I believe, as well established as the primary truths in any department of human knowledge.

Application 1. To make a straightedge. Having formed a ruler as straight as possible and drawn a line with it upon a plane surface, turn the ruler

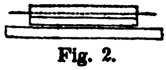

Fig. 2.

* When the word *line* is used alone, *straight* line is to be understood.

over upon the opposite side, and, with the same edge, repeat the line, coinciding with, or better, very nearly coinciding with the first; if the lines coincide, or appear equally distant throughout, the edge is sensibly straight.

Application 2. To test the parallelism of the edges of the ruler, place it against a second straightedge, and, having drawn a line, turn it end for end and draw a second. (Why ?)

Application 3. To test a plane, apply the straightedge to it in different directions.

Def. 6. When two lines intersect each other, or would intersect if sufficiently produced, the inclination of the one line to the other is called an *Angle*. It is obvious that angles are of different and comparable magnitudes, and, therefore, like other geometrical quantities, solids, surfaces, and lines, are capable of addition, subtraction, multiplication, and division, and, consequently, subject to mathematical investigation.

PROPOSITION II.

The sum of the adjacent angles formed by one line meeting another, is always the same constant quantity. (72)

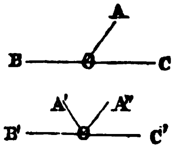

Fig. 3.

Let the line AO meet the line BOC in O, making the angle AOB any whatever, and, in like manner, let the line A′O meet the line B′OC′ in O, making the angle A′OB′ any whatever. Placing the first figure upon the second, let the point O coincide with the point O and the line OB take the direction OB′ then will OC take the direction OC′ (69) and the line BOC will coincide with B′OC′, also OA will take a certain direction OA″, which we are at liberty to suppose between OA′ and OC′; whence the angle AOB will be equal to the angle A″OB′, the two becoming identical, and AOC to A″OC′; but the angle A″OB′ is, by hypothesis, the sum of the angles A′OB′ and A′OA″, and therefore greater than A′OB′ by A′OA″ (1_2), while the angle A″OC′ is less than A′OC′ by the same quantity; therefore the sum of the angles A″OB′, A″OC′ is equal to the sum of the angles A′OB′, A′OC′ (1_{11}), whence the sum of the angles AOB, AOC, is equal to the sum of the angles A′OB′, A′OC′ (1_7) which was to be proved.

The same in the language of symbols;

Let BOC coincide with B′OC′, and OA take the direction OA″;

$\therefore \quad \angle\, AOB = A''OB' = A'OB' + A'OA''$

and $\quad \angle\, AOC = A''OC' = A'OC' - A'OA''$;

$\therefore \quad \angle\, AOB + AOC = A'OB' + A'OC'$. Q. E. D.*

The method employed in the preceding demonstration is obviously that of *superposition*, and the principles upon which it is grounded are the nature of the straight line, the whole is equal to the sum of its parts, and, equals added to equals, the sums are equal.

Def. 7. If the angle $AOB = AOC$, then AOB and AOC are called ***Right Angles;*** hence, AOB or AOC being the half of $AOB + AOC =$ a constant quantity (72), is also a constant quantity, or,

Fig. 32.

Cor. 1. All right angles are equal to each other. (73)

Cor. 2. The sum of the adjacent angles formed by one line meeting another is equivalent to two right angles. (74)

Cor. 3. Conversely; two lines met by a third, so as to make the sum of the adjacent angles equal to two right angles, form one and the same straight line. (75)

For if not, let BOX be a straight line, while

$\angle\, AOB + AOC = \perp$,† [by hypothesis;]

then (74) $\quad \angle\, AOB + AOX = \perp$;

$\therefore$ subtracting $AOX - AOC = 0$,

therefore BOC does not differ from the straight line BOX.

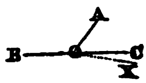

Fig. 33.

Cor. 4. The vertical angles formed by intersecting lines are equal. (76)

If a and b be vertical angles, while c is adjacent to both, we have (74) $\quad a + c = \perp$,

and $\quad b + c = \perp$;

$\therefore \quad a + c = b + c$,

and (1_{11}) $\quad a = b$.

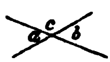

Fig. 34.

Cor. 5. The sum of all the angles that can be formed at a given point and on the same side of a straight line, is equal to two right angles (2). (77)


Fig. 35.

Cor. 6. The sum of all the angles that can be formed round a given point, is equivalent to four right angles. (78)


Fig. 36.

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* "*Quod erat demonstrandum,*" which was to be proved.

† We shall use the symbol ⊥ for two right angles, and + for four, while ∟ will signify a single right angle.

Application 1. To make a *Rightangle* for drawing perpendiculars. Form a triangle, having one angle as nearly right as possible; set its base upon a straight edge and draw a line by its perpendicular, turn the triangle over upon the opposite face, and repeat the line. The instrument may be three inches in base and six in altitude, and cut from pasteboard, brass, or ivory, or framed of wood.

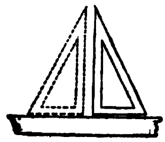


Fig. 4.

Application 2. To make a right angle in the field. Take a piece of board about 10 inches square, bore a hole through the centre, and fit to it a staff $4\frac{1}{2}$ feet long, so that, when the staff is stuck vertically in the ground, the board shall turn freely upon its top and continue during a complete revolution in the same plane, which will be determined by sighting its upper surface at a given object; draw two lines at right angles to each other through the centre and from corner to corner. In these lines at the corners may be stuck four needles—and to know whether they are accurately at right angles, we have only to place the board as nearly as possible in a horizontal position, and then to sight at two marks in range with the needles, in the lines drawn at right angles, and to see if we hit the same marks when the board is turned a quarter round. Why?

Fig. 5.

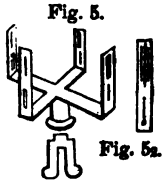


Fig. 5_2.

Fig. 5_3.

The SURVEYOR'S CROSS is the same instrument, only better finished.

The sight vanes (*fig.* 5_2) are hairs opposite slits, and *vice versâ;* and the piece that turns upon the staff is represented in *fig.* 5_3.

To survey with the cross, it is necessary to be provided with two straight flag-staffs, which should be wound with a red flag; or they may be square, and the adjoining faces painted red and white. In running a line, one staff is to go before and the other is to be left at the last position of the cross for a back sight. A measuring rod, tape line, or *Gunter's chain* is also to be provided. The chain is 4 rods or 66 feet in length, and centesimally divided by a hundred links, each, consequently, equal to 7·92 inches. It is a maxim in surveying land that all instruments, whether for measuring lines or angles, must be kept in a horizontal position; for it is the base, or the *projection* of the field upon the same horizontal plane that is required.

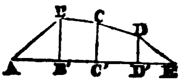


Fig. 6.

Let it be required to survey the field ABCDE. We set the cross at B′ in the line AE so as to make BB′A a right angle, and in the same way we find the points C′, D′; we then measure the lines AB′, B′C′, C′D′, D′E, and the *offsets* B′B, C′C, D′D, when the field is surveyed. This is a good method of determining an irregular boundary, ABCDE, as that of a river. Frequently it will be better to take the offsets from a diagonal, AC.

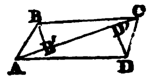


Fig. 6a.

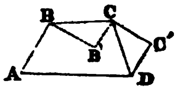


Fig. 6a.

Sometimes it will be preferable, especially if the corners cannot be seen from each other, to measure AB then BB′ at right angles to BA, then B′C at right angles to B′B, then CC′ at right angles to CB′, and finally C′D at right angles to C′C. The student should now make an actual survey and draw an accurate plot of it on paper. This may be done by aid of the ruler, rightangle, and a line of equal parts, which are to be run off of any convenient magnitude by a pair of dividers.

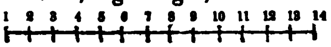


Fig. 7.

PROPOSITION III.

Two lines intersected by a third, making the alternate angles equal to each other, are parallel. (79)

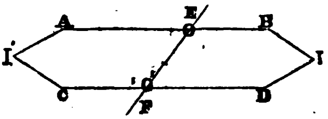


Fig. 8.

Let the lines AB, CD be intersected by EF in O and O′, making $\angle$ AOO′ = OO′D; then will AB, CD, be parallel. If not, let AB, CD, produced, meet on the right in I, then it may be shown that they will meet on the left in a point I′, and consequently that two straight lines, IBAI′, IDCI′, may intersect in two points, which is contrary to (67).

For, since $\angle$ BOO′ + AOO′ = $\perp$ = CO′O + DO′O,
and by hypothesis, $\angle$ AOO′ = DO′O,
$\therefore$ subtracting, $\angle$ BOO′ = CO′O;

from which it follows that the figure BOO′D may be reversed and applied to the figure CO′OA; for the line OO′ being reversed and applied in O′O, the line O′D will take the direction OA and therefore coincide with OA through its whole extent, and, for a like reason, OB will coincide with O′C.

Hence, if the lines OB, O'D, intersected in I, the lines OA, O'C, now coinciding with them in their reversed position, would have the same point of intersection in I'.

Cor. 1. Conversely, two parallel lines intersected by a third, make the alternate angles equal. (80)

For, if the lines AB, CD, being parallel, do not make the angles AOO', OO'D equal, draw A'OB' (the student will draw the line) making $\angle$ A'OO' = OO'D, then will (79) A'B' be parallel to CD, but, by hypothesis, AB is parallel to CD, $\therefore$ through the same point O two lines AB, A'B', have been drawn parallel to the same line CD, which is contrary to (71). [*fig.* 8.]

Cor. 2. The equality of two alternate angles determines: (81)

1°. The equality of all the other alternate angles, whether internal or external. (Where is this shown?)

2°. The equality of the external to the opposite internal on the same side of the secant line. (Why?)

3°. The sum of the internal angles, or the sum of the external, on the same side of the secant line, to be equal to two right angles.

For, if the angle DO'E = AOF, adding $\angle$ BOF to both sides, we have

$$\angle \text{DO'E} + \text{BOF} = \text{AOF} + \text{BOF} = \perp.$$

Cor. 3. If the secant line be perpendicular to one of two parallel lines, it will be perpendicular to the other also. (82)

Cor. 4. Two lines parallel to the same line, are parallel to each other. [How can this be shown from (82)?] (83)

Cor. 5. Two angles having their sides parallel, and lying in the same direction, are equal. For, producing the sides of the angles a, b, till they meet, forming the angle c, we have (81, 2°), $a = c = b$. (84)

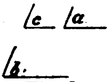


Fig. 8$_2$.

Cor. 6. The opposite angles of a *Parallelogram* are equal. (85)

It is scarcely necessary to remark that a parallelogram is a quadrilateral, or four-sided figure, having its opposite sides parallel.

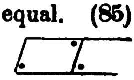


Fig. 8$_3$.

Application. Draw parallel lines upon paper with the straightedge and rightangle, and construct them in the field by aid of the cross.

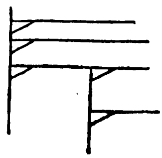


Fig. 8.

PROPOSITION IV.

The sum of the external angles of a Polygon, formed by producing the sides outward, is equal to four right angles. (86)

Let a, b, c, d, e, be the external angles of a polygon; through any point, either within or out of the polygon, draw lines parallel to the sides of the polygon, forming the angles a', b', c', d', e', corresponding severally to the angles a, b, c, d, e; then (84) we have

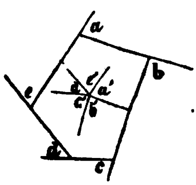


Fig. 9.

$$\angle a = a',\ b = b',\ c = c',\ d = d',\ e = e';$$

$\therefore$ adding, $a + b + c + d + e = a' + b' + c' + d' + e' = +$. Q. E. D.

If we denote the corresponding internal angles by A, B, C, ..., of which we will suppose there are n, we shall have (74),

Fig. 9a.

$$A + a = \perp,\ B + b = \perp,\ C + c = \perp,\ \ldots [n]$$

$$\therefore \quad A + B + C + \ldots + a + b + c + \ldots = n(\perp),$$

but (86) $\quad a + b + c + \ldots = 2(\perp);$

$\therefore$ subtracting, $A + B + C + \ldots [n] = (n - 2)(\perp)$. $\therefore$

Cor. 1. The sum of the angles of a polygon is equal to two right angles taken as many times as there are sides less two. (87)

Making successively $n = 6, 5, 4, 3$, we have

$$(n - 2)(\perp) = 4(\perp),\ 3(\perp),\ 2(\perp),\ (\perp);\ \text{therefore}$$

Cor. 2. The sum of the angles of a *Hexagon* is equal to eight right angles. (88)

Cor. 3. The sum of the angles of a *Pentagon* is equal to the sum of six right angles. (89)

Cor. 4. The sum of the angles of a *Quadrilateral* is equivalent to four right angles. $\therefore$ (90)

Cor. 5. All the angles of a quadrilateral may be right angles; such a figure is called a *Rectangle.* (91)

Cor. 6. The sum of the angles of a *Triangle* is equal to two right angles. (92)

Cor. 7. The sum of the acute angles of a *Right angled Triangle* is equal to a right angle (93)

Cor. 8. The external angle formed by producing one of the sides of a triangle, is equal to the sum of the two opposite internal angles. Let a, b, c, be the three angles of any triangle, and c adjacent to the external angle d; we have (94)

Fig. 93.

$$\angle\, a+b+c=\perp,$$

and (?) $\qquad d+c=\perp;$

$\therefore$ (?) $\qquad \angle\, d=a+b.$

Scholium. The angles of a triangle will be determined:

1°. When the three angles are equal; (how?)

2°. When two angles are equal and the third known; (how?)

3°. When two angles are given. (How?)

What is a hexagon? pentagon? quadrilateral? triangle? right angled triangle?

EXERCISES.

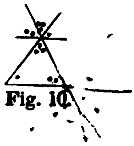


Fig. 10.

1°. By aid of the straightedge and rightangle, construct about the vertex of a triangle angles equal to those at the base, and do this for triangles of different forms.

2°. Through the vertex of any triangle draw a line parallel to the base and prove (92).

3°. The same by drawing a parallel to one of the oblique sides.

4°. The same by drawing perpendiculars to the base.

5°. From any angle of a polygon, draw diagonals to all the other angles and prove (87).

6°. From any point within a polygon, draw lines to the several angles and prove the same.


Fig. 11.

7°. Let fall a perpendicular from the right angle of a right angled triangle upon the hypothenuse, and prove that the three triangles thus formed are equiangular.


Fig. 12.

8°. Having drawn the sides of two angles respectively perpendicular to each other, prove that the angles are either equal or that one is what the other wants of two right angles.


Fig. 13.

9°. Prove that two triangles are mutually equiangular when the sides of the one are severally perpendicular to the sides of the other.

SECTION SECOND.

Equal Polygons.—First Relations of Lines and Angles.

PROPOSITION I.

If, in two Polygons, excepting three parts which are (95)
adjacent, viz., two angles and an included side, or two sides and an included angle, the remaining parts, taken in the same order, are severally equal, the Polygons will be equal throughout, and the excepted parts will be equal, each to each.

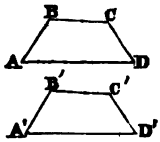
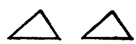


Fig. 14.

First. Let AB = A'B', ∠ B = B', BC = B'C', ∠ C = C', CD = C'D': then, applying the figure ABCD to A'B'C'D', let the point A fall on A' and the line AB take the direction A'B'; the point B will fall on B', since AB = A'B' by hypothesis, and BC will take the direction B'C', because the angle B = B', by hypothesis, and the point C will fall on C'; (why?) and finally, the line CD will coincide with C'D', (why?) so that the points A and D coinciding with A' and D', the line AD will coincide with A'D', and the polygons will coincide throughout, and ∠ A will = A', AD = A'D', D = D'.

Second. Let ∠ A = A', AB = A'B', ∠ B = B', BC = B'C', ∠ C = C'; then by a superposition altogether similar to that above, it may be made to appear that the polygons will coincide throughout, and that the three adjacent parts, CD, ∠ D, DA, will be severally equal to the three, C'D', ∠ D', D'A'; and it is obvious that the same reasoning may be extended to polygons of any number of sides. Q. E. D.

Cor. 1. If two triangles have two sides and the included (96)
angle of the one, equal to the two sides and included angle of the other, each to each, the triangles will be equal, and the remaining three parts of the one respectively equal to the remaining three parts of the other.



[The student should letter the figure and explain.] Fig. 14_2

Cor. 2. If two triangles have two angles and the included (97)
side of the one, severally equal to the two angles and the included

side of the other, the triangles will be equal throughout. [By what figure, and how, may this be illustrated ?]

Cor. 3. The diagonal divides the parallelogram into two equal triangles ; .·. (98)

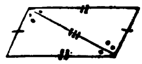


Fig. 14_3.

1°. The opposite sides of a parallelogram are equal.

2°. Parallels are everywhere equally distant.

[Letter the figure and explain carefully. What condition must be imposed upon the parallelogram in order to illustrate the last part of the corollary ?]

Cor. 4. The diagonals of a parallelogram mutually bisect, or divide each other into equal parts (98, 1°), (80), (97). (99)

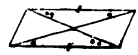


Fig. 14_4.

Cor. 5. An *Isosceles* triangle is bisected by the line which bisects the angle embraced by the equal sides. (100)

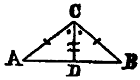


Fig. 14_5.

For let $CA = CB$ and $\angle\ ACD = BCD$;

then (96) will $\triangle\ ACD = BCD$; .·.

Cor. 6. In an isosceles triangle, the angles opposite the equal sides are themselves equal ; as $\angle\ A = B$, and (101)

Cor. 7. The line bisecting the vertical angle of an isosceles triangle bisects the base, to which it is also at right angles or perpendicular. (102)

Cor. 8. Equilateral triangles are equiangular (100). (103)

PROPOSITION II.

In any triangle, that angle is the greater which is opposite the greater side. (104)

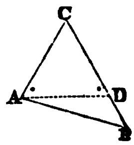


Fig. 15.

Let CB be $>$ CA ; then will $\angle$ CAB $>$ CBA. For, taking CD, a part of CB, equal to CA and joining AD, we have

$\angle\ CAB > CAD = CDA = ABD + BAD > ABD.$

Q. E. D.

[Give the reasons, Consult (?), (101). (94), (?).]

Cor. 1. Conversely, the side opposite the greater angle, is the greater. For if the side opposite the greater angle be not the greater, it must be equal or less ; it cannot be equal, for then would the angles be equal (?), neither can it be less, for then would the opposite angle be the less (104), which is contrary to the hypothesis. (105)

Cor. 2. A triangle having two equal angles, is isosceles (106)
(105).

Cor. 3. An equiangular triangle is equilateral (106). (107)

What is an *Isosceles* triangle? What an *Equilateral* triangle?

PROPOSITION III.

Any two sides of a triangle are together greater than (108)
the third.

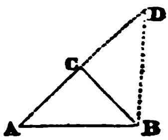


Fig. 16.

Let ABC be any triangle; then will the sum of any two sides, as AC + CB, be > the third AB.

Produce AC to D, making CD = CB, and join DB.

then (101) $\angle CDB = CBD < ABD$;

$\therefore$ (105) $AB < AD = AC + CD = AC + CB$.

Q. E. D.

Cor. 1. The sum of the lines joining any point within a (109)
triangle and the extremities of one of its sides, is less than the sum of the other two sides.

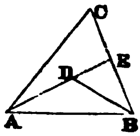


Fig. 16$_2$.

Let D be any point in the triangle ABC; produce AD to meet CB in E, then

$AD + DB < AD + DE + EB < AC + CE + EB$.

(Why?) Q. E. D.

Cor. 2. Any side of a polygon is less than the sum of all (110)
the other sides.

[Letter and show how.]


Fig. 16$_3$.

$\therefore$ *Cor.* 3. Of two polygons, standing upon the same (111)
base and one embracing the other, the perimeter of the enveloping is greater than that of the enveloped.

[See the last figure and explain.]

Cor. 4. The straight line is the shortest that can be (112)
drawn from one point to another.

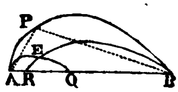


Fig. 16$_4$.

First. Let APB be any line, whether curved or broken, but concave toward the straight line AB, then APB will be greater than AB. For, taking any point P, in the line APB, and revolv-

ing the branches AP, BP, about A and B till the point P coincides with the straight line AB, first in Q, then in R, the straight lines AP, BP, in the positions AQ, BR, will overlap each other by a certain line RQ, since the st. line AP + st. l. BP > AB—and therefore, the branches AP, BP, falling on the same side of AB, since, by hypothesis, the path APB is concave toward AB, will intersect each other in some point E; whence the path APB, being equal to the sum of the paths AE + EQ and RE + EB, will be longer than the path AEB by the sum of the parts RE and QE. The same may be shown of any other path that is concave toward AB except AB.

Second. Let the path A*cdefgh*B be any whatever, having flexures either continuous or abrupt in *c*, *d*, *e*, *f*, *g*, *h*; then drawing the straight lines A*c*, *cd*, *de*, *ef*, *fg*, *gh*, *h*B,

Fig. 16_5.

we have (110) AB < polygonal perimeter (A*c* + *cd* + *de* + *ef* + *fg* + *gh* + *h*B) < the path (A*cdefgh*B), as shown above, since all the parts of A*cdefgh*B are concave toward the several sides of the polygon. Q. E. D.

Cor. 5. "Hence also, we may infer, that of any two (113) paths, ACB, ADB, leading from A to B, and everywhere concave toward the straight line AB, that which is enveloped by the other, as ADB, is the shorter. For of all the paths not lying between ADB and the straight line AB, there is none, ADB excepted, than which a shorter may not be found. And this is the case whether the paths ACB, ADB, be both of them curvilineal, or one of them, (ACB or ADB), rectilineal." *

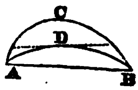


Fig. 16_6.

Cor. 6. The perpendicular is the shortest distance from (114) a point to a straight line; and of oblique lines, that is the shorter which is nearer the perpendicular—and those equally distant are equal.

For let PA be perpendicular to CBAB′, and AB′ = AB < AC, and produce PA in Q, making AQ = AP; joining QB, QB′, QC, we have

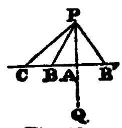


Fig. 16_7.

$$PC + QC > PB + QB > PA + QA \quad (?)$$

$\therefore$ dividing by 2 $\quad PC > PB > PA, \quad (?)$

also $\quad PB = PB'. \quad (?)$

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* Library of Useful Knowledge, Geometry.

Cor. 7. All points equally distant from the extremities (115) of a straight line are situated in the same perpendicular passing through the middle of that line.

The points C, B, A, B′, are equally distant from P and Q.

PROPOSITION IV.

If two triangles have two sides of the one equal to two (116) *sides of the other, each to each, but the included angles unequal, then the third sides will be unequal, and the greater side will be that opposite the greater angle.*

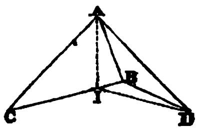


Fig. 17.

Let two of the equal sides be made common in AB, and the other two, AC, AD lie on opposite sides of AB; draw AI bisecting the angle CAD and terminating in I, which will be a point in CB on the hypothesis that CAB is the greater angle, since then we have $\angle$ CAB $>$ CAI $=$ DAI $>$ DAB;—finally, join ID. We have (96), (108),

$$BC = BI + IC = BI + ID > BD. \quad \text{Q. E. D.}$$

Cor. 1. Conversely, if two triangles have two sides of (117) the one equal to two sides of the other, each to each, but the third sides unequal, the opposite angles will be unequal and that will be the greater which is opposite the greater side. For it can be neither equal nor less. (Why?)

Cor. 2. If two triangles have the sides of the one severally equal to the sides of the other, the angles opposite the equal sides will be severally equal. (118)

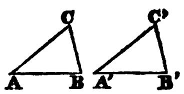


Fig. 17₂.

For, if AB = A′B′, BC = B′C′, AC = A′C′; $\therefore \angle$ C cannot be $\gtrless$ C′, for then would AB $\gtrless$ A′B′; $\therefore \angle$ C = C′, and, for a like reason, $\angle$ A = A′, B = B′.

Cor. 3. When the opposite sides of a quadrilateral are (119) equal, each to each, the figure will be a parallelogram [*fig.* 14₃].

EXERCISES.

1°. It is a principle in Optics that a ray of light, AP, impinging upon a reflecting surface, MM′, makes the angle of reflection BPP′, equal to the angle of incidence APP′, PP′ being perpendicular to MM′ at P. Given in position the luminous point A, the eye, B, and the mirror, MM′, to find the point P from which the light is reflected to B, and to prove that APB is the shortest path from A to B by way of the mirror, or that $AP + PB < AQ + QB$.

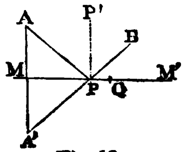


Fig. 18.

2°. Having given in position the elastic planes PQ, QR, and the points A, B; it is required to find the track of an elastic ball projected from A upon PQ and rebounding from QR so as to hit a pin at B.

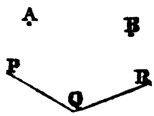


Fig. 18$_2$

3°. Given in position the straight line MN and the points A, B; required the point O in MN such that AO shall = BO.

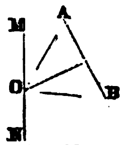


Fig. 19.

4°. Prove that the perpendiculars drawn through the middle points of the three sides of a triangle, will intersect in the same point.

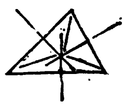


Fig. 20.

5°. Having given in position the lines AB, CD, and the point P, it is required to draw through P a line *mn*, which shall make equal angles with AB, CD.

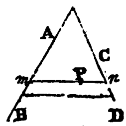


Fig. 21.

6°. Let AOB be a right angle, COB an equilateral triangle, and OP perpendicular to CB. Prove that the angles AOC, COP, POB are all equal. How could you trisect a right angle?

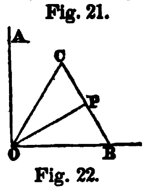


Fig. 22.

7°. Let A, B, two angles of an equilateral triangle, be bisected by the lines AO, BO, and from O draw OP, OQ, parallel to the sides CA, CB, and terminating in the base in P and Q. Prove that $AP = PQ = QB$. [The student will construct the figure.]

8°. Prove that any side of a triangle is greater than the difference of the other two.

9°. Prove that the angle included between two lines drawn from the vertex of any triangle, the one bisecting the vertical angle and the other perpendicular to the base, is half the difference of the basic angles.

10°. Prove that if a line be drawn from one of the equal angles of an isosceles triangle upon the opposite side and equal to it, the angle embraced by this line and the production of the base, or unequal side, will be three times one of the equal angles of the triangle.

11°. Draw three lines from the acute angles of a right angled triangle—two bisecting these angles and the third a perpendicular to one of the bisecting lines—and prove that the triangle embraced by these lines will be isosceles.

SECTION THIRD.

Proportional Lines.

PROPOSITION I.

The segments of lines intercepted by parallels are proportional. (120)

In the first place, let the secant line AC be divided into commensurable parts by the parallels AA′, BB′, CC′, for instance, such that AB containing three measures, BC shall contain two of the same; it may be shown that any other secant line, A′B′C′ will be divided in the same ratio.

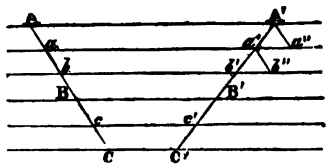


Fig. 23.

For, dividing AB, BC, into 3 and 2 parts in a, b, c, and drawing aa', bb', cc', through the points of division parallel to AA′ and through A′, a', b', &c. (points of A′C′), drawing A′a'', $a'b''$, &c., parallel to AC and terminating severally in the parallels aa', bb', &c., we have (?) (?),

$A'a'' = Aa = ab = a'b''$,

and $\angle\, a''A'a' = b''a'b'$, $\angle\, a''a'A' = b''b'a'$;

$\therefore$ $A'a' = a'b'$, and for a like reason, $a'b' = b'B' = B'c' = c'C'$.

$\therefore$ AB : BC : : 3 : 2, and A′B′ : B′C′ : : 3 : 2.

$\therefore$ AB : BC : : A′B′ : B′C′.

The same reasoning will be applicable, it is obvious, whatever may be the number of parts into which AB, BC, may be divided, or whenever AB, BC are commensurable.

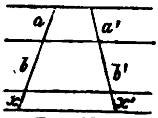


Fig. 23₂.

Next, let the segments a, b, be incommensurable. Produce b so that $b+x$ shall be commensurable witn a, and through the extremity of x draw a parallel increasing the corresponding segment b' by x'; then will $b'+x'$ be commensurable with a'; and, by what has already been demonstrated, we shall have the proportion

$$\frac{b+x}{a} = \frac{b'+x'}{a'}, \text{ or } \frac{b}{a} + \frac{x}{a} = \frac{b'}{a'} + \frac{x'}{a'};$$

but x can be made less than any assignable quantity, since it is less than the measure of a, which may be diminished at pleasure; whence it follows that $\frac{x}{a}$, $\frac{x'}{a'}$, are variable quantities capable of indefinite diminution, and, consequently (63), that

$$\frac{b}{a} = \frac{b'}{a'} \text{ or } a : b :: a' : b'. \quad \text{Q. E. D.}$$

Cor. 1. When three lines, two of which are parallel, are cut by two others so as to make the segments of the secant lines proportional, the third line is parallel to the other two. (121)

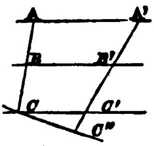


Fig. 23₃.

For, if the line CC′ be not parallel to AA′, let CC″ be so; then there will result (120)

$$AB : BC :: A'B' : B'C'',$$

$$\therefore \quad B'C'' = \frac{BC \times A'B'}{AB},$$

but, by hyp., AB : BC : : A′B′ : B′C′, or $B'C' = \dfrac{BC \times A'B'}{AB}$;

$\therefore$ B′C″ = B′C′, which is absurd; therefore CC′ is not otherwise than parallel to AA′.

Cor. 2. The sides of mutually equiangular triangles are proportional. (122)

For, let any two of the equal angles be made vertical, then will

the bases m, n, be parallel, and, if through the common vertex a line be drawn parallel to m, n, dividing their intercepted perpendicular into the parts a'', b'', we shall find (120)

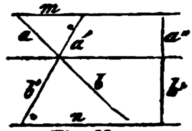


Fig. 23 .

$$a : b :: a'' : b'' :: a' : b'. \quad \text{Q. E. D.}$$

Remark. The homologous sides are opposite equal angles.

Cor. 3. If two triangles have an angle of the one equal to an angle of the other and the sides about the equal angles proportional, the triangles will be mutually equiangular and consequently similar. Draw a line through the extremity of b [*fig.* 23_4], see (122), and imitate the reasoning under (121). (123)

Def. Figures which have their angles respectively equal when taken in the same order, and their sides proportional, also taken in the same order, are called similar.

Cor. 4. The altitudes of similar triangles are to each other as their bases. For we have (122) (124)

$$a'' : b'' :: a : b :: m : n.$$

Cor. 5. If lines be drawn from the vertex of a triangle to points in the base, all lines parallel to the base will be divided into parts proportional to the segments of the base. (125)

For the segments ab, bc, cd, have to AB, BC, CD, the constant ratio of the perpendiculars Op, OP.

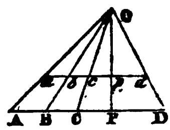


Fig. 23_5.

Cor. 6. Triangles having their sides severally parallel are similar. (126)

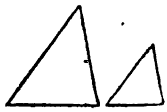


Fig. 23_6.

Cor. 7. Triangles which have their sides respectively perpendicular, are similar. Thus, a, b, c, are to each other as a', b', c', where it is obvious that the homologous sides are those that are perpendicular to each other. (127)

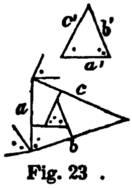


Fig. 23 .

PROPOSITION II.

A right angled triangle, by a perpendicular let fall from the right angle, is divided into two partial triangles, similar to itself, and consequently to each other. (128)

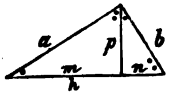


Fig. 24.

Let a, b, denote the sides about the right angle, h the side opposite, or the hypothenuse, and m, n, the segments of h made by the perpendicular, p. The angle (a, h) is common to the triangles (a, b, h), (a, m, p), which also have a right angle in each, and are therefore equiangular, ($\angle\,(b, h)$ being $= \angle\,(p, a)$,) and consequently similar. So it may be shown that $\Delta\,(b, n, p)$ is similar to $\Delta\,(h, b, a)$, and the proposition is demonstrated.

Cor. 1. The perpendicular is a mean proportional between the segments of the hypothenuse. (129)

For we have $\qquad m : p :: p : n.$

Cor. 2. The sides about the right angle are mean proportionals between the hypothenuse and the adjacent segments. For we have (122) (130)

$$\frac{h}{a} = \frac{a}{m}, \text{ and } \frac{h}{b} = \frac{b}{n}.$$

Cor. 3. The segments of the hypothenuse are to each other as the squares of the sides opposite them; for, from the above we have (131)

$$hm = a^2,$$

and $\qquad hn = b^2;$

$\therefore$ by div., $\qquad m : n = a^2 : b^2.$

Cor. 4. The square of the hypothenuse is equal to the sum of the squares of the other two sides. For we have (132)

$\therefore \qquad (m + n)h = a^2 + b^2,$

or $\qquad hh = h^2 = a^2 + b^2.$

Scholium. This proposition, the celebrated 47th of Euclid, is obviously a direct consequence of (122), and with that theorem, which embraces the characteristic property of the triangle, constitutes the working rules of all geometry.

Cor. 5. Conversely, if the square of one side of a triangle is equal to the sum of the squares of the other two, the triangle is right angled. (133)

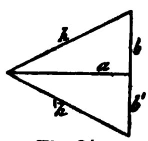


Fig. 24a.

For suppose the triangle (h, a, b) to be such that $h^2 = a^2 + b^2$; then if on a we construct the triangle (h', a, b') making $b' = b$ and the $\angle\,(a, b') = \llcorner$, there will result $h'^2 = a^2 + b'^2 = a^2 + b^2 = h^2$, $\therefore h' = h$, and

$\therefore \qquad \angle(a, b) = (a, b') = \llcorner.$

Cor. 6. The diagonal of a square is incommensurable with its side, the former being to the latter as $\sqrt{2}$ to 1. (134)

Fig. 24 a.

For we have $x^2 = a^2 + a^2 = 2a^2$

$\therefore$ $x = a\sqrt{2}$, or $x : a = \sqrt{2} : 1.$

The student might very naturally suppose that a common measure to any two lines could be found by taking the measuring unit sufficiently small; but, from this corollary, it is shown that the contrary may be true, and we are therefore taught the necessity of extending our demonstrations to the case of incommensurability.

PROPOSITION III.

To find the relation between the oblique sides, a, b, *of a triangle, the line* c *drawn from the vertex to the base, and the segments,* m, n, *of the base made by this line.*

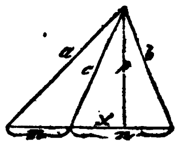


Fig. 25

Drop the perpendicular p, intercepting the segment x between the foot of p and that of c; we have (132), (8), (9),

$$a^2 = p^2 + (m + x)^2 = p^2 + m^2 + 2mx + x^2,$$
$$b^2 = p^2 + (n - x)^2 = p^2 + n^2 - 2nx + x^2,$$
$$c^2 = p^2 + x^2;$$

$\therefore$ subtracting the 3d from the 1st, we have

$$a^2 - c^2 = m^2 + 2mx,$$

from the 2d $\quad b^2 - c^2 = n^2 - 2nx;$

$\therefore$ dividing one by m and the other by n,

$$\frac{a^2 - c^2}{m} = m + 2x,$$

$$\frac{b^2 - c^2}{n} = n - 2x,$$

$$\therefore \quad \frac{a^2 - c^2}{m} + \frac{b^2 - c^2}{n} = m + n; \text{ i. e.,} \qquad (135)$$

If in any triangle, a line be drawn from one of the angles, terminating in the opposite side, and the squares of the sides embracing the divided angle be severally diminished by the square of the dividing line, then the sum of the quotients arising from dividing these remainders by the corresponding subjacent segments, will be equal to the divided side.

If we take the particular case in which m and n are equal, there will result

$$\frac{a^2-c^2}{m}+\frac{b^2-c^2}{m}=m+m,$$

whence $$a^2+b^2=2c^2+2m^2, \text{ or,}$$

Cor. 1. If a line be drawn from the vertex of a triangle (136) to the middle of the base, the sum of the squares of the oblique sides will be double the sum of the squares of the middle line and the half base.

If the triangle be isosceles, there will result

$$\frac{a^2-c^2}{m}+\frac{a^2-c^2}{n}=m+n,$$

or $$a^2-c^2=mn.$$

Cor. 2. In an isosceles triangle, the square of the oblique (137) side diminished by the square of the middle line, is equal to the product of the segments of the base.

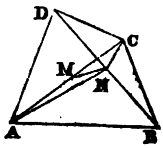


Fig. 25₂.

Let ABCD be any quadrilateral, M, N, the middle points of its diagonals, AC, BD; join AN, NC; then (136)

$$AB^2+AD^2=2AN^2+2BN^2,$$

and $$CB^2+CD^2=2CN^2+2BN^2;$$

$$\begin{aligned}\therefore\quad AB^2+BC^2+CD^2+DA^2 &= 4BN^2+2(AN^2+CN^2)\\ &= 4BN^2+2(2AM^2+2MN^2)\\ &= (2AM)^2+(2BN)^2+4MN^2\\ &= AC^2+BD^2+4MN^2.\end{aligned}$$

$\therefore$ *Cor.* 3. The sum of the squares of the sides of a (138) quadrilateral, exceeds the sum of the squares of its diagonals by four times the square of the line joining the middle points of the diagonals.

Cor. 4. The sum of the squares of the sides of a parallelogram, is equal to the sum of the squares of its diagonals. (139)

For (99) the diagonals bisect each other.

PROPOSITION IV.

To find the distance of the foot of the perpendicular of a triangle from the middle of its base.

Let a, b, be the oblique sides of a triangle, and p its perpendicular, dividing the base $2m$ into two parts, $m+x$, $m-x$;

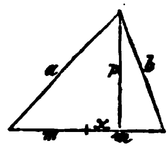


Fig. 25 a.

where, consequently, x is the distance of the foot of the perpendicular to the middle point of the base. We have (132)

$$a^2 = p^2 + (m + x)^2$$

and $$b^2 = p^2 + (m - x)^2 ;$$

$\therefore$ $$a^2 - b^2 = 4mx,$$

or (10), $$(a + b)(a - b) = 2m \cdot 2x ;$$

$\therefore$ (43) $$2m : a + b = a - b : 2x.$$

The base of any triangle is to the sum of the oblique sides as their difference to the double distance of the foot of the perpendicular from the middle point of the base. (140)

This theorem is convenient for finding the perpendicular let fall upon any side of a triangle; for we have

$$p = [a^2 - (m + x)^2]^{\frac{1}{2}} = [(a + m + x)(a - m - x)]^{\frac{1}{2}} \qquad (141)$$

PROPOSITION V.

If a line be drawn bisecting the vertical angle of a triangle, the segments of the base thus formed, will be to each other as the sides opposite them. (142)

In figure 16, draw CE parallel to DB, E being a point in AB; then (81), (80), [*fig.* 16]

$$\angle \text{ACE} = \text{CDB} = \text{CBD} = \text{BCE},$$

and (120) $$\text{AE} : \text{EB} :: \text{AC} : \text{CD} = \text{CB}. \quad \text{Q. E. D.}$$

EXERCISES.

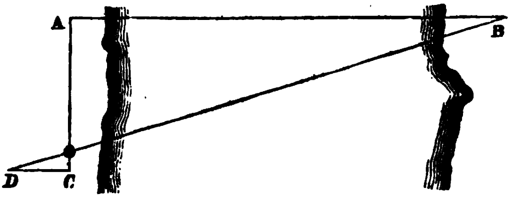


Fig. 26.

1°. Wishing to ascertain the distance AB of an inaccessible object B, I measure a line AC at right angles to AB, equal to 12 chains, then I take CD back and at right angles to AC, equal to 5 chains; and find that a line sighted from D to B intersects AC at O, distant from C 3·25 chains. What is the distance from A to B?

We have $AB : AO = CD : CO$,

or $$\frac{x}{12 - 3{\cdot}25} = \frac{5}{3{\cdot}25}, \therefore\ x = \frac{5 \cdot 8{\cdot}75}{3{\cdot}25} = 13{\cdot}46.$$

How can the same thing be done by (129)?

2°. What must be the length of a ladder to turn between two buildings, one 20, the other 30 feet high, and 40 feet apart; and what must be the distance of its foot from the first-mentioned building?

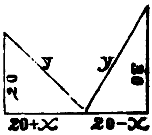


Fig. 27.

Let the length of the ladder be denoted by y, and the distances of its foot from the buildings by $20 + x$ and $20 - x$; then

$$(20 + x)^2 + 20^2 = y^2 = (20 - x)^2 + 30^2,$$

or $$20^2 + 2 \cdot 20x + x^2 + 20^2 = 20^2 - 2 \cdot 20x + x^2 + 30^2;$$

$$\therefore\ 4 \cdot 20x = 30^2 - 20^2 = (30 + 20)(30 - 20),$$

and $$x = \frac{50 \cdot 10}{80} = \frac{50}{8} = 6{\cdot}25;$$

and the distance required is $20 + x = 26{\cdot}25$;

$$\therefore\ \text{length of ladder} = y = (20^2 + 26{\cdot}25^2)^{\frac{1}{2}} = ?$$

3°. Wishing to ascertain the distance from A to D (*fig.* 6_2), rendered inaccessible by the intervention of a ledge of rocks, I fetch a compass, ABB′CC′D, viz., AB = 11 chs., BB′ = 8, B′C = 3, CC′ = 4, and C′D = 5. What is the distance from A to D?

Ans. 15 chs.

4°. Given the perpendiculars let fall from A and B [*fig.* 18] upon the plane MM′, the one 7 the other 5 feet, and the distance of these perpendiculars = 10 feet; to find the distance of P from the greater perpendicular. *Ans.* 5 feet 10 in.

5°. Given $AB = a$ and the distances of A and B from [*fig.* 19] MN, $= p, p'$; to find $AO = x$.

6°. Given the hypothenuse of a right angled triangle, h, and the sum of the sides about the right angle, $= 2m$; to find these sides.

Denote them by $m + x$, $m - x$;

Fig. 28.

then (132) $x = \pm (\frac{1}{2}h^2 - m^2)^{\frac{1}{2}}$;

$$\therefore\ m + x = m \pm (\tfrac{1}{2}h^2 - m^2)^{\frac{1}{2}},$$


Fig. 28_2.

and $m - x = m \mp (\frac{1}{2}h^2 - m^2)^{\frac{1}{2}}$;—where it is obvious that both values given by the double sign $\pm$ equally satisfy the conditions of the problem.

7°. Given the hypothenuse of a right angled triangle and the difference of the sides about the right angle, to determine the triangle.

If the hypothenuse be denoted by $2a$ and the difference of the legs by $2m$, the answers will be

$$x+m=\pm(2a^2-m^2)^{\frac{1}{2}}+m,\ x-m=\pm(2a^2-m^2)^{\frac{1}{2}}-m.$$

8°. Given the base of a right angled triangle, b, and the sum of the hypothenuse and perpendicular $2m$, to find these sides.

9°. Given the base of a right angled triangle and the difference of the hypothenuse and perpendicular, to determine the triangle.

10°. A liberty-pole 100 feet long was broken, and, resting upon the stump, its top fell at the distance of 40 feet. Required the length of a ladder, planted 10 feet distant, that shall reach the break.

11°. A greyhound is 10 rods distant from a hare; the hare starts off at right angles to the line joining them and the hound pursues in the hypothenuse of a right angled triangle to intercept her. Now the velocity of the dog is to that of the hare as 3 to 2. How far does each run?

12°. The same conditions as in the preceding only that the hare runs obliquely from the hound at half a right angle.

Ans. The hound runs $60\,(\sqrt{2}+\sqrt{2{\cdot}05})$,
the hare $40\,(\sqrt{2}+\sqrt{2{\cdot}05})$, rods.

13°. Given the perimeter $4a$, of a rectangle (right angled parallelogram) and the diagonal, d, to determine the figure.

14° Given the perimeter, na, of a right angled triangle, and the perpendicular, a, let fall from the right angle on the hypothenuse, to determine the triangle.

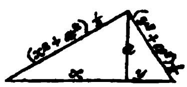


Fig. 29.

Let the segments of the hypothenuse be represented by x and y; then shall we have

$$x+y+(x^2+a^2)^{\frac{1}{2}}+(y^2+a^2)^{\frac{1}{2}}=na,$$

and $\dfrac{x}{a}=\dfrac{a}{y}$, which ratio may be put $=z$;

then x will $=az$ and $y=\dfrac{a}{z}$,

whence $az+\dfrac{a}{z}+(a^2z^2+a^2)^{\frac{1}{2}}+\left(\dfrac{a^2}{z^2}+a^2\right)^{\frac{1}{2}}=na,$

or $z+\dfrac{1}{z}+(z^2+1)^{\frac{1}{2}}+\dfrac{1}{z}(z^2+1)^{\frac{1}{2}}=n;$

$$\therefore \quad \left(1+\frac{1}{z}\right)(z^2+1)^{\frac{1}{2}} = n - z - \frac{1}{z},$$

and $z^2+2z+1+1+\frac{2}{z}+\frac{1}{z^2} = n^2-2nz-\frac{2n}{z}+z^2+1+1+\frac{1}{z^2}$,

or $$(2n+2)\,z+\frac{2n+2}{z} = n^2,$$

$$\therefore \quad z^2-\frac{n^2}{2n+2}\cdot z = -1,$$

$$\therefore (58),\ z^2-\frac{n^2}{2n+2}\cdot z+\frac{n^4}{(4n+4)^2} = \frac{n^4}{16(n+1)^2}-1$$

$$= \frac{n^4-16\,(n+1)^2}{16\,(n+1)^2},$$

$$\therefore \quad z-\frac{n^2}{4n+4} = \pm\frac{[n^4-16\,(n+1)^2]^{\frac{1}{2}}}{4\,(n+1)},$$

or, $$z = \frac{n^2\pm[n^4-16(n+1)^2]^{\frac{1}{2}}}{4\,(n+1)} = \frac{n^2\pm\{[n^2+4(n+1)]\cdot[n^2-4(n+1)]\}^{\frac{1}{2}}}{4\,(n+1)};$$

$$\therefore \quad x = a\cdot\frac{n^2\pm\{[n^2+4\,(n+1)]\cdot[n^2-4\,(n+1)]\}^{\frac{1}{2}}}{4\,(n+1)},$$

and $$y = \frac{a}{\dfrac{n^2\pm\{[n^2+4\,(n+1)]\cdot[n^2-4\,(n+1)]\}^{\frac{1}{2}}}{4\,(n+1)}}.$$

15°. Given the two lines, m, n, drawn from the acute angles of a right angled triangle to the middle points of the opposite sides, to determine the triangle.

16°. In a triangle, given the segments a, b, of the base formed by the perpendicular let fall from the vertex, and the ratio, $m : n$, of the oblique sides, to determine the triangle.

Ans. $m\left[\frac{(a+b)\,(a-b)}{(m+n)\,(m-n)}\right]^{\frac{1}{2}}$, $n\left[\frac{(a+b)\,(a-b)}{(m+n)\,(m-n)}\right]^{\frac{1}{2}}$, oblique sides.

17°. In a triangle, given the base, b, the perpendicular, p, and the difference, $2m$, of the oblique sides, $x+m$, $x-m$, to determine these sides. *Ans.* $x+m = ?$ $x-m = ?$

18°. In a triangle, given the segments, a, b, of the base made

by the line bisecting the vertical angle, and the sum, $2m$, of the oblique sides, to determine the triangle.

$$m+x=\frac{2am}{a+b},\ m-x=\frac{2bm}{a+b}.$$

19°. The same as in the preceding, only the difference of the oblique sides is given instead of their sum.

20°. Given the three sides of a triangle, 12, 18, and 20 chains, to find the perpendicular let fall on the last-named side from the opposite angle. *Ans.* 10·66 chains.

21°. Given the base, b, and altitude, h, of a triangle, to determine the inscribed rectangle, the base of which shall be to the altitude as m to n. *Ans.* The sides of the rectangle are $\frac{mbh}{hm+bn}$, $\frac{nbh}{hm+bn}$.

22°. The same as the preceding, except that the sum of the sides containing the rectangle is given.

23°. The same, only the difference is given.

24°. Being on the bank of a river and wishing to ascertain its breadth, for the want of instruments I set up a stick at A and then take 25 measures with a rod happening to be at hand, from A in a straight line to B, where I set up a second stick; I then take 5 measures of the same rod directly back from A in line with X, an object on the opposite bank, to C; also from B backward in line with X, 5 measures to D; and finally I find BC to be $=25$ measures and a remainder, also AD $=26+$ a remainder. Now to determine these remainders, I proceed as follows: For the fractional part of BC, I apply to the measuring rod, m, a second rod, r, $=$ the excess of BC over $25m$, and find r contained in m once, with a remainder, r'; therefore $\frac{m}{r}=1+\frac{r'}{r}$. Again, applying r' to r, I find it contained once with a remainder, r''; $\therefore\ \frac{r}{r'}=1+\frac{r''}{r'}$. Applying r'' to r', I find it contained 4 times with a remainder, r'''; $\therefore\ \frac{r'}{r''}=4+\frac{r'''}{r''}$. Finally, I find r''' contained in r'' just twice; $\therefore\ \frac{r''}{r'''}=2$.

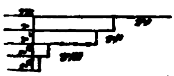


Fig. 30.

Whence $$\frac{r}{m}=\frac{1}{1+\frac{r'}{r}}=\frac{1}{1+\frac{1}{\frac{r}{r'}}}=\frac{1}{1+\frac{1}{1+\frac{r''}{r'}}}$$

$$=\frac{1}{1+\frac{1}{1+\frac{1}{\frac{r'}{r''}}}}=\frac{1}{1+\frac{1}{1+\frac{1}{4+\frac{r'''}{r''}}}}=\frac{1}{1+\frac{1}{1+\frac{1}{4+\frac{1}{\frac{r''}{r'''}}}}}$$

or $$\frac{r}{m}=\frac{1}{1+\frac{1}{1+\frac{1}{4+\frac{1}{2}}}}=\frac{1}{1+\frac{1}{1+\frac{2}{9}}}=\frac{1}{1+\frac{9}{11}}=\frac{11}{20};$$

$$\therefore \quad BC=25{\cdot}55;$$

by a like process I find $$AD=26+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{6}}}}=26{\cdot}65.$$

We have then (135) putting AX $=x$, BX $=y$,

$$\frac{25{\cdot}55^2-25^2}{5}+\frac{y^2-25^2}{x}=5+x,$$

and $$\frac{26{\cdot}65^2-25^2}{5}+\frac{x^2-25^2}{y}=5+y;$$

$\therefore \quad x= \quad , y= \quad .$

25°. Given the hypothenuse of a right angled triangle $=35$ rods, and the side of the inscribed square $=12$ rods, to determine the triangle. *Ans.* base $=28$, or $=21$;
perpendicular $=21$, or $=28$.

SECTION FOURTH.

Comparison of Plane Figures.

PROPOSITION I.

Rectangles are to each other as the products of their dimensions. (143)

We will take the general case at once, and suppose the containing sides, a, a', of the rectangle A are incommensurable with b, b', those of B. Let b be increased by x, and b' by x', so as to make $b+x$ commensurable with a, and $b'+x'$ with a'. Suppose, for instance, that while a is divided into m parts, that $b+x$ contains n of the same parts, or that (2$_4$)

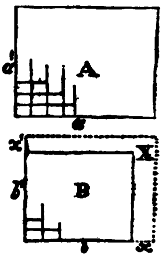


Fig. 31.

$$\frac{a}{b+x}=\frac{m}{n};$$

and that, while a' is divided into m' equal parts, which may be different from those of a, $b'+x'$ contains n' of the same parts, or that

$$\frac{a'}{b'+x'}=\frac{m'}{n'}.$$

Through the points of division draw lines parallel to the sides of the rectangles and denote the part added to B by X. Then will the partial rectangles be all equal (95); and the number constructed on the base a being = to m, and the number of tiers corresponding to the divisions in a', = n', we have mm' for the total number of partial rectangles in A; so the rectangle B + X contains nn' of the same. Hence (2$_4$)

$$A : B+X = mm' : nn';$$

but $$mm' : nn' : : aa' : (b+x)(b'+x'),$$

since a, a'; $b+x$, $b'+x'$, are as their measures m, m'; n, n';

whence $$\frac{B+X}{A}=\frac{(b+x)(b'+x')}{aa'}=\frac{bb'+bx'+b'x+xx'}{aa'},$$

or $$\frac{B}{A}+\frac{X}{A}=\frac{bb'}{aa'}+\frac{bx'+b'x+xx'}{aa'};$$

$\therefore$ (63) $$\frac{B}{A}=\frac{bb'}{aa'}.\quad \text{Q. E. D.}$$

For it is obvious that X depends for its value upon x, x', decreasing as these decrease so as to become nothing if these were to become $= 0$, and that x, x', being less than the parts into which a, a' are divided, may be made less than any assignable quantity, by sufficiently increasing the points of division. Therefore

$$\frac{X}{A} \text{ and } \frac{bx' + b'x + xx'}{aa'},$$

may be made less than any assignable quantity.

Cor. 1. The rectangle is measured by the product of its dimensions. For suppose A to become a measuring square, or that while $A = 1$ is taken as the unit of surface, $a = a' = 1$ becomes the linear unit, we have (144)

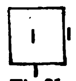


Fig. 31a.

$$\frac{B}{1} = \frac{bb'}{1 \cdot 1}, \text{ or } B = bb'.$$

Cor. 2. The right angled triangle is measured by half the product of its base into its altitude. Thus $T = \frac{1}{2}bh$. (145)

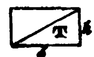


Fig. 31b.

PROPOSITION II.

The Trapezoid is measured by half the sum of its parallel sides multiplied into their perpendicular distance. (146)

Let a, b, be the parallel sides and from their extremities drop the perpendiculars h, h, on a, b, produced, forming the rectangle

$$X + Trp + Y,$$

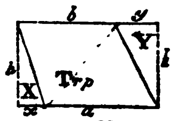


Fig. 32.

composed of the trapezoid Trp and the right angled triangles X, Y, having the bases x, y. We have (144), (145),

$$X + Trp + Y = (x + a) \cdot h = \tfrac{1}{2} \cdot 2\,(x + a) \cdot h$$
$$= \tfrac{1}{2}(x + a + b + y) \cdot h,$$

and $\quad X \qquad + Y = \frac{1}{2}xh + \frac{1}{2}yh;$

$\therefore \quad Trp = \frac{1}{2}(a + b) \cdot h.$ Q. E. D.

Cor. 1. The parallelogram is measured by the product of its base into its altitude. For, when $b = a$, the trapezoid becomes a trapezium or parallelogram, (147)

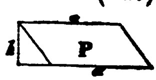


Fig. 32a.

and $\qquad Trp = \frac{1}{2}(a + b) \cdot h,$

becomes, $\qquad P = \frac{1}{2}(a + a) \cdot h = ah.$

Cor. 2. The triangle is measured by half the product of its base and altitude. (148)

For $Trp = \frac{1}{2}(a+b) \cdot h$,

becomes $T = \frac{1}{2}ah$.

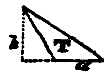


Fig. 32_3.

Cor. 3. Parallelograms are to each other, and triangles are to each other, as the product of their bases and altitudes. (149)

Cor. 4. Parallelograms of the same or equal altitudes are to each other as their bases. (150)

Cor. 5. Triangles of the same or equal altitudes are to each other as their bases. (151)


Fig. 32_4.

Cor. 6. Parallelograms of the same or equal bases are to each other as their altitudes. (152)

Cor. 7. Triangles of equal bases are to each other as their altitudes. (153)

Cor. 8. Parallelograms of equal bases and altitudes, are equivalent. (154)


Fig. 32_5.

Cor. 9. Triangles of equal bases and altitudes, are equivalent. (155)


Fig. 32_6.

Cor. 10. Parallelograms, or triangles, between the same parallels, and on the same or equal bases, are equivalent. (156)

Cor. 11. Conversely, if parallelograms or triangles, standing on equal bases in the same straight line, and having their vertices turned in the same direction, be equivalent, the line of vertices will be parallel to the line of bases. (157)

For if the triangles, having the bases a, a, and the perpendiculars p, p', be equivalent, there results

$$\tfrac{1}{2}ap = \tfrac{1}{2}ap',$$

$$\therefore \quad p = p';$$

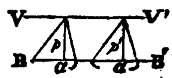


Fig. 32_7.

whence the line of vertices VV′ is parallel to the line of bases BB′.

Cor. 12. The parallelogram is double the triangle of the same base and altitude. (158)

[Compare (147) with (148).]

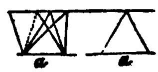


Fig. 32_8.

PROPOSITION III.

Two triangles, having an angle of the one equal to an angle of the other, are to each other as the products of the sides about the equal angles. (159)

Let the equal angles of the triangles A, B, be made vertical, and join the extremities of the sides a, b, forming the triangle X; then (151)

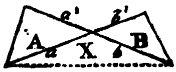


Fig. 33.

$$\frac{A}{X}=\frac{a'}{b};$$

and
$$\frac{X}{B}=\frac{a}{b'};$$

$\therefore$
$$\frac{A}{B}=\frac{aa'}{bb'}.\quad \text{Q. E. D.}$$

Cor. 1. Equiangular parallelograms are to each other as the products of their dimensions (158). (159_2)

Cor. 2. Similar triangles are to each other as the squares of their homologous sides. For, then we have (122) (160)

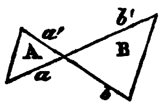


Fig. 33_2.

$$\frac{a'}{b'}=\frac{a}{b};$$

$\therefore$
$$\frac{A}{B}=\frac{aa'}{bb'}=\frac{a}{b}\cdot\frac{a'}{b'}=\frac{a}{b}\cdot\frac{a}{b}=\frac{a^2}{b^2}.$$

Cor. 3. If figures, resolvable into the same number of similar triangles, be constructed upon the three sides of a right angled triangle, that on the hypothenuse will be equal to the sum of those described upon the other two sides. For we have (161)

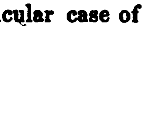


Fig. 33_3.

$$\frac{B}{A}=\frac{b^2}{a^2},\text{ and }\frac{C}{A}=\frac{c^2}{a^2};\ \therefore\ \frac{B+C}{A}=\frac{b^2+c^2}{a^2}=\frac{a^2}{a^2}=1.$$

Scholium. It is obvious that (132) is but a particular case of this more general theorem.

EXERCISES.

1°. Demonstrate (132) by turning two squares into one.

Fig. 34.

2°. On the oblique sides of any triangle describe parallelo-

grams, any whatever; and on the base, construct a parallelogram having its sides parallel to the line drawn through the vertex and the intersection formed by producing the sides of the parallelograms described on the oblique sides of the triangle, and terminating in these same sides.

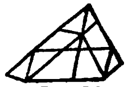


Fig. 35.

Prove that the parallelogram on the base of the triangle will be equal to the sum of the other two; and that, if the vertical angle become right, and the parallelograms squares, (132) will be proved.

3°. It is required to straighten the line ABC, separating the estates of two gentlemen, by aid of the cross alone.

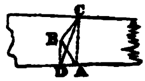


Fig. 36.

Draw BD parallel to CA, then DC will be the required line. (Why?)

4°. Straighten the line in figure 36_2.

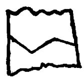


Fig. 36_2.

5°. It is required to change the direction of the line AB, so that the extremity A shall be at A′, still retaining the same amount of land on each side.

Fig. 36_3.

6°. To turn a quadrilateral into a triangle, either by the cross in the field, or by the right-angle straightedge on paper.


Fig. 36_4.

7°. To turn a pentagon into a triangle.


Fig. 36_5.

8°. To turn a triangle into a rectangle.


Fig. 36_6.

9°. To turn a rectangle, A, into a square, B.

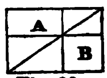


Fig. 36_7.

10°. Draw any polygon on paper, and find a square that shall be equal to it in area.

11°. There is a well at P, in the side of a triangular field; it is required to draw a line from P so as to divide the field into two equal parts.

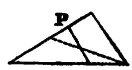


Fig. 36_8.

12°. What is the area of a parallelogram, 20 chains in length and 15 in breadth? *Ans.* 30 acres.

13°. The longer side of a rectangle is r times the shorter, and

the area is $= m^2$, or to a square having m for its side. What are its dimensions?

We have $rx \cdot x = m^2$,

$$\left.\begin{aligned} \therefore \quad x &= \frac{m}{\sqrt{r}} = \frac{m \cdot \sqrt{r}}{r}, \\ \text{and} \quad rx &= m\sqrt{r}. \end{aligned}\right\} \textit{Ans.}$$

14°. A surveyor would lay out a rectangular field of 20 acres, such that the length may be three times the breadth. Required, the dimensions. [Solve and prove.]

15°. The area of a parallelogram being represented by m^2, and the base by b, how shall we find the height, h?—given h, how shall we find b? *Ans.* $h = \frac{m^2}{b}$, $b = \frac{m^2}{h}$. How expressed in words?

16°. A man has 10 acres to put in a parallelogram, having one side 12 chains. How wide must it be? *Ans.* 8 chs. $33\frac{1}{3}$ links.

17°. If the area of a triangle be m^2, and its base b, how shall we find its height, h? *Ans.* $h = \frac{2m^2}{b}$. [Enounce in words.]

18°. A surveyor would lay out a triangle of one hundred acres on a base of 25 chains. Required, the altitude.

19°. A joiner has a board 10 feet long, 2 feet wide at one end, and 2 feet 6 inches at the other. How many square feet in the board? (146) *Ans.* $22\frac{1}{2}$ feet.

20°. Given the sides of a triangle, 15, 16, 17 chs. Required, the area.

Taking 16 for base, we have (140)

$$16 : 17+15 :: 17-15 : 2x,$$

or $\quad 8 : 32 :: 1 : 2x, \therefore = 4;$

$$\therefore (141)\ p = [(17+8+2)(17-8-2)]^{\frac{1}{2}} = [9 \cdot 3 \cdot 7]^{\frac{1}{2}} = 3\sqrt{21}$$

$$\therefore \quad \text{Area} = 8p = 24\sqrt{21} \text{ sq. chs.} = 11 - \tfrac{1}{500} \text{ acres, nearly.}$$

21°. What is the area of a triangle, the sides of which are 13, 19, 34 chs.?

What difficulty arises in attempting to solve this problem? and why?

22°. Wishing to ascertain the area of the quadrilateral field ABCD (*fig.* 6_3), I avail myself of the measures in the third exercise of the preceding section, and find it to be nine acres and one-fifth.

23°. In order to obtain the area of a pentagon ABCDE, I mea-

sure the sides BC, CD, DE, and find them BC = 6, CD = 4, DE = 10 chains; and, by aid of the cross, determine the perpendiculars AP = 3, AP_2 = 8, AP_3 = 11·8, let fall upon CB, DC, DE, produced. The area will be found to be eight acres and two-fifths.

24°. Given the sides of a triangle ABC,

$$AB = 30,\ AC = 40,\ BC = 50 \text{ chs.},$$

to cut off a triangle ABD (D being a point in AC) equal to 45 acres.

We find the area of the triangle ABC = 60 acres; therefore (151)

$$AD = 30 \text{ chs., the line required.}$$

25°. The triangle being the same as in the above, it is required to cut off the 45 acres by a line B′C′ parallel to BC—B′ being a point in AB, and C′ a point in AC.

We find (160)

$$AB' = 25{\cdot}98,\ AC' = 34{\cdot}64 \text{ chs.}$$

26°. The same conditions as in the preceding, except that 30 acres are to be cut off by a line PQ, P being a point in AB, 20 chs. distant from A. Required AQ, measured off from A towards C.

Join PC, then

30 chs. : 20 chs. : : 60 acres : APC, ∴ = 40 acres;
and 40 acres : 30 acres : : 40 chs. : AQ, ∴ = 30 chs.

27°. To draw a line through a given point in the plane of a given angle so as to include a given area.

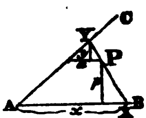


Fig. 37.

Let the given point P be embraced by the sides of the given angle CAB. Suppose the problem solved, and that YPX is the required line: we are to find x = AX, a portion of AB. As the position of the point P, in regard to the lines AB, AC, is given, we may suppose the perpendicular p, let fall from P upon x and a, the distance of P from the line AC, measured parallel to AB, to be known. Further, let the area of the triangle AXY be denoted by m^2, that is, by a square whose side is m.

We have, y being a perpendicular let fall from Y upon a,

$$x(y+p) = 2m^2,$$

and

$$\frac{x}{a} = \frac{y+p}{y} = 1 + \frac{p}{y},$$

or

$$\frac{p}{y} = \frac{x}{a} - 1 = \frac{x-a}{a},$$

$$\therefore \quad \frac{y}{p} = \frac{a}{x-a}, \ \therefore\ y = \frac{pa}{x-a};$$

$$\therefore \quad x(y+p) = x\left(\frac{pa}{x-a}+p\right) = \frac{pax}{x-a}+px = 2m^2,$$

or $$pax + px^2 - pax = 2m^2x - 2m^2a,$$

$$\therefore \quad x^2 - \frac{2m^2}{p} \cdot x = -\frac{2m^2a}{p},$$

$$\therefore \quad x^2 - 2 \cdot \frac{m^2}{p} \cdot x + \left(\frac{m^2}{p}\right)^2 = \frac{m^4}{p^2} - \frac{2m^2a}{p} = \frac{m^2}{p^2}(m^2 - 2pa),$$

$$\therefore \quad x - \frac{m^2}{p} = \pm \frac{m}{p}(m^2 - 2pa)^{\frac{1}{2}},$$

and $$x = \frac{m^2}{p} \pm \frac{m}{p}(m^2 - 2pa)^{\frac{1}{2}} = \frac{m}{p}[m \pm (m^2 - 2pa)^{\frac{1}{2}}].$$

Since $(m^2 - 2pa)^{\frac{1}{2}}$ is $< m$, the two values of x,

$$x = \frac{m}{p}[m + (m^2 - 2pa)^{\frac{1}{2}}],$$

and $$x = \frac{m}{p}[m - (m^2 - 2pa)^{\frac{1}{2}}],$$

are both plus; and, therefore, equally applicable to the problem, as shown in *fig.* 37_2, where we have

$$x = AX,\ x = AX'.$$

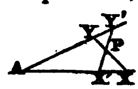


Fig. 37_2.

We observe here since

the tr. AXY $= m^2 =$ AX′Y′,

that subtracting the quadrilateral AX′PY from both, there results the triangle XPX′ = YPY′.

But if we take the difference of the values of X, we have

$$XX' = AX - AX' = \frac{m}{p} \cdot 2(m^2 - 2pa)^{\frac{1}{2}},$$

which is the solution of the following problem:

$27°_2$. Through a given point, P, in the side of a given triangle, AXY, to draw a line X′Y′, terminating in AX, AY, produced, if necessary, so that the areas PXX′, PYY′, shall be equal.

We remark that m^2 must be at least as great as $2pa$; for, were $m^2 < 2pa$, $(m^2 - 2pa)^{\frac{1}{2}}$ would be *imaginary* (6_3), and the values of x, $x =$ AX, $x =$ AX′ unreal, or the problem impossible. That is, the area to be cut off cannot be greater than the double inscribed rectangle standing upon a.

We have here an example of a quantity capable of a *minimum*, or *least value ;* and, in order to find this minimum, it is obvious that we have only to solve the quadratic and put the part under the radical equal to zero.

Thus $$m^2 - 2pa = 0,$$
gives $$m^2 = 2pa, \text{ for the } \textit{minimum} \text{ of } m^2.$$

If we inquire what is the greatest value of m^2 in this problem, we shall find it infinite, or that m^2 has no *maximum*. [See the value of x and *fig.* 37_3.]

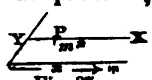


Fig. 37_3.

If now we subtract from a by insensible degrees, or, as it is commonly expressed, diminish a according to the LAW OF CONTINUITY, a will at length become $= 0$,

when $$x = \frac{m}{p}\,[m \pm (m^2 - 2pa)^{\frac{1}{2}}],$$

gives $$x = \frac{m}{p}\,[m \pm m] = \frac{2m^2}{p}, \text{ or } = 0,$$

Fig. 37_4.

where the first value only is applicable.

Continuing the same motion, a will evidently become *minus*, and the point P will at the same time take up its position on the left of the line AC, or without the angle, and we shall have

$$x = \frac{m}{p}\,[m \pm (m^2 + 2pa)^{\frac{1}{2}}],$$

Fig. 37_5.

where both values of x are applicable, only that the second, being minus, requires the area m^2 to be laid off on the left.

If, in like manner, we make p pass through the value 0 it will change its sign, and P will take up a position below the line AB,

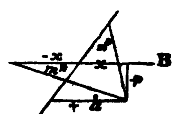


Fig. 37_6.

whence $$x = \frac{m}{p}\,[m \pm (m^2 - 2pa)^{\frac{1}{2}}],$$

becomes $$x = \frac{m}{-p}\,[m \pm (m^2 - 2 \cdot -pa)^{\frac{1}{2}}]$$

or $$x = \frac{m}{p}\,[-m \mp (m^2 + 2pa)^{\frac{1}{2}}],$$

where both values are applicable, as indicated in the figure.

If, at the same time, we make a and p both minus, there will result

$$x = \frac{m}{-p}\,[m \pm (m^2 - 2 \cdot -p \cdot -a)^{\frac{1}{2}}],$$

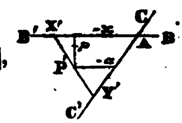


Fig. 37_7.

$$\text{or} \qquad x = -\frac{m}{p}\,[m \pm (m^2 - 2pa)^{\frac{1}{2}}],$$

where we have the same values as in the original solution, only changing $+x$ into $-x$, as we evidently ought to do, since x is measured from A in an opposite direction. This is obvious also from the comparison of the figures 37_7 and 37_8 where all the parts of the one correspond to all the parts of the other. The figure 37_7 will have the same double construction that 37 has in 37_2.

What would be the result of making $p = 0$? what, when $p = 0$, $a = 0$?

The problem we have just been discussing is admirably adapted to show the correlation of algebraical signs and geometrical figures. This problem will also enable us to divide any polygon into any required parts.

28°. The parallel sides of a trapezoid are a, b, and the altitude, h, it is required to cut off an area m^2 adjacent to b by a line parallel thereto. What will be its breadth?

$$Ans.\ x = -\frac{bh}{a-b} \pm \left[\frac{2hm^2}{a-b} + \left(\frac{bh}{a-b}\right)^2\right]^{\frac{1}{2}}.$$

29°. Given the sides of a triangle, ABC, viz., AB = 10 chains, AC = 8 chains 43 links, BC = 4·70; also the position of a point, P, viz., distant from AB by 1·80, and from AC by one chain. Required to draw a line through the point P, that shall divide the triangle into two equal parts.

The student will solve and verify.

30°. What is the greatest rectangle with a given perimeter?

We may denote the sides by $a + x$ and $a - x$, then will the perimeter be $= 4a$, and we shall find

$$x = \pm (a^2 - m^2)^{\frac{1}{2}},$$

m^2 denoting the area; whence it is obvious that m^2 cannot exceed a^2; and therefore that $x = 0$, when the area m^2 is a maximum, or that among rectangles of the same perimeter, the square is the maximum.

31°. What is the maximum triangle that can be constructed on a given base, and of a given perimeter?

Ans. The triangle must be *isosceles.*

32°. Given the area and diagonal of a rectangle, to determine its sides.

33°. Let ABCD be a quadrilateral field, of which the sides AB,

BC, CD, DA, are, respectively, 16, 34, 30, 20 rods in length; also, let the diagonal BD = $37\frac{1}{4}$ rods. It is required to divide the field into two equal parts, by a line cutting the opposite sides, AB, CD, so that the ratio of the segments of the one shall be equal to that of the corresponding segments of the other.

Ans. AX = ? DY = ?

[Produce AB, CD, until they meet, and consult (140), (141), (132), (159).]

BOOK THIRD.

PLANE GEOMETRY DEPENDING ON THE CIRCLE, ELLIPSE, HYPERBOLA, AND PARABOLA.

SECTION FIRST.

The Circle.

Definition 1. The circle is a plane figure described by the revolution of a straight line of invariable length about one of its extremities as a fixed point.

Def. 2. The describing line is called the *Radius* [*rod*] of the circle, the fixed point the *Centre*, and the curve line that bounds it its *Circumference*.

Cor. All radii, or lines drawn from the centre to the circumference, are equal to each other. (162)

PROPOSITION I.

Angles at the centre of the same or equal circles, are to each other as their subtending arcs; and the corresponding sectors have a like ratio. (163)

In the first place, let the angles be commensurable. For example, suppose the angle AOB to contain the angle BOC twice; then it is manifest that in applying the angle BOC twice to the angle AOB, the point C will fall on C′, the middle point of the arc AB, since (162) OC′ = OC. Therefore the arc AB contains the arc BC twice; and, in the same way, the sector AOB is double the sector BOC.

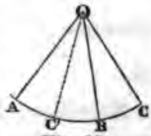


Fig. 38.

∴ ∠ AOB : ∠BOC : : arc AB : arc BC : : sec. AOB : sec. BOC, each being as 2 to 1.

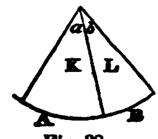


Fig. 38_2.

So in general, if a, b, be any commensurable angles, or such that when a is divided into any number, m, of equal parts, b shall also be exactly divisible into some number, n, of the same equal parts; then it will follow, by superposing one of these equal angles m times upon a and n times upon b, that the corresponding arcs A and B will be divided into m and n equal arcs. The same of the sectors K, L.

$\therefore \quad a : b :: A : B :: K : L$, being as m to n.

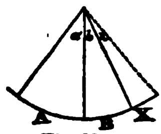


Fig. 38_3.

Finally, let us take the most general case, or that of incommensurability, where a and b *having no common* measure, are incapable of being divided into the same equal parts. Let b be increased by the angle x, so that $b+x$ shall be commensurable with a, then will the corresponding arc $\mathrm{B}+\mathrm{X}$ be commensurable with A; and, from what has just been proved, there results the proportion

$$\frac{b+x}{a}=\frac{B+X}{A},$$

or

$$\frac{b}{a}+\frac{x}{a}=\frac{B}{A}+\frac{X}{A};$$

$$\therefore (63) \quad \frac{b}{a}=\frac{B}{A}, \text{ or } \frac{a}{b}=\frac{A}{B}, \text{ or } a : b :: A : B,$$

and the same is equally true of the sectors. Q. E. D.

Cor. 1. In the same or equal circles, arcs may be taken as the *measures* of their angles at the centre. (164)

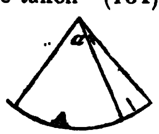


Fig. 38_4.

Thus, if b be taken for the unit of angles and B for that of arcs,

the proportion $$\frac{a}{b}=\frac{A}{B},$$

becomes $$\frac{a}{1}=\frac{A}{1}$$

or $$a=A.$$

Scholium. As the right angle seems the most suitable for comparing angles, so its measure, the *Quadrant*, or quarter circumference, would appear to be the appropriate unit of arcs, and for this purpose the French have sometimes employed it, dividing the quadrant into a hundred equal parts, which they called degrees: but custom has established a different unit, the $\frac{1}{90}$th part of the quad-

rant, which is denominated a degree, and written 1°. The degree is divided into 60 minutes, marked 1′ and the minute into 60 seconds, written 1″, 2″, 3″, ...

Cor. 2. A quarter circumference is the measure of a right angle, a semicircumference of two right angles, and a circumference of four right angles. (165)

How many degrees in half a quadrant? in one third of a quadrant? in $\frac{1}{4}$? $\frac{1}{5}$? $\frac{1}{6}$? $\frac{1}{7}$? $\frac{1}{8}$? $\frac{1}{9}$? $\frac{1}{10}$?

Cor. 3. In the same or equal circles, the greater arc subtends the greater angle at the centre, and, conversely, the greater angle is subtended by the greater arc. (166)

Cor. 4. In the same or equal circles, equal arcs subtend equal angles at the centre, and the converse. (167)

Cor. 5. In the same or equal circles, the greater arc subtends the greater chord, and conversely, the greater chord is subtended by the greater arc. (168)

For, let the arc B > A, ∴ (166) $\angle b > a$; ∴ (116) chord $d > c$. Conversely, let chord $d > c$, ∴ $\angle b > a$; ∴ arc B > A.

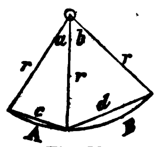


Fig. 38_5.

Cor. 6. In the same or equal circles, equal arcs subtend equal chords, and the converse. (169)

Cor. 7. The *Diameter*, or chord passing through the centre, is the greatest straight line that can be drawn in a circle. Why? (170)

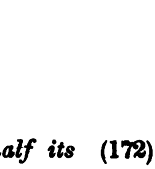


Fig. 38_6.

Cor. 8. The diameter bisects the circle, and its circumference. (171)

PROPOSITION II.

An angle inscribed in a circle, is measured by half its subtending arc. (172)

Let AOB be any diameter, and C any point of the circumference: join CA, CO, then (94), (162), (101),

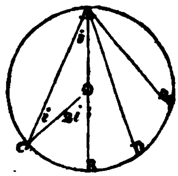


Fig. 39.

$\angle$ COB = OAC + OCA = 2 $\angle$ OAC;

$\therefore$ (164), measure of $\angle$ OAC = meas. of $\frac{1}{2}\angle$ COB

= $\frac{1}{2}$ arc CB;

so meas. of $\angle$ DAB = $\frac{1}{2}$ arc DB,

and meas. of $\angle$ EAB = $\frac{1}{2}$ arc EB;

$\therefore$, by addition and substraction,

meas. of $\angle$ CAD = $\frac{1}{2}$ arc CD, [Centre where?]

and meas. of $\angle$ DAE = $\frac{1}{2}$ arc DE. [!] Q. E. D.

Cor. 1. In the same or equal circles, angles at the circumference, subtended by the same or equal arcs, are equal. (173)

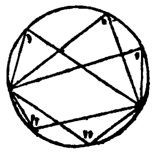
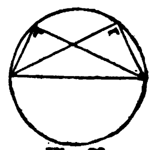


Fig. 39_2.

Cor. 2. Of angles inscribed in the same or equal circles: (174)

1°. An angle subtended by an arc less than a semicircumference is less than a right angle; (*fig.* 39_2.)

2°. An angle subtended by an arc greater than a semicircumference is greater than a right angle; (*fig.* 39_2.)



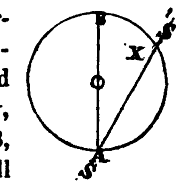
3°. An angle subtended by a semicircumference is a right angle.

Fig. 39_3.

Cor. 3. The sum of the opposite angles of a quadrilateral inscribed in a circle is equal to two right angles. (175) (*fig.* 39_2.)

$\cdot + : = \perp.$

Cor. 4. A *Secant* line is always oblique to the diameter drawn through either point, in which it cuts the circumference; and, conversely, a line drawn through the extremity of a diameter, and oblique to it, is a secant line, or cuts the circle. (176)



For, let SS′ be any secant line, cutting the circumference in any points A, X, and draw the diameter AB; then the $\angle$ BAX is < ∟, being measured by the arc BX < semicircumference. Conversely, let the line SAS′ be oblique to the diameter AB, and suppose that BAS′ is the acute angle, then will SS′ cut the circle. For, take the point X in the

Fig. 39_4.

semicircumference on the same side of the diameter AB with the angle BAS′, so that the angle BAX shall be equal to the angle BAS′, which may always be done, since, by taking the point X nearer and nearer to A, the angle BAX may be made any whatever less than a right angle; then will the line AX coincide with the line AS′, and the point X will be a point of the line AS′. Therefore AS′ will cut the circumference a second time in X, and be secant to the circle.

Cor. 5. A *Tangent*, that is a line which touches a circle (177) without cutting it, is at right angles to the diameter drawn through the point of tangency, and the converse.

Let TAT′ be tangent to the circle at A; then will TT′ be perpendicular to the diameter AB; for, if TAT′ were oblique to AB, it would be a secant line (176). Again, let TAT′ be perpendicular to AB, then is TT′ tangent to the circle; for, if AT were a secant, then would $\angle$ BAT $<$ ∟, which is contrary to the hypothesis.

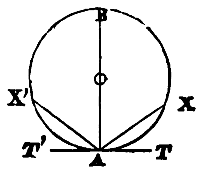


Fig. 39_5.

Illustration. Imagine a line to revolve about a point in the circumference of a circle, there will be but one position in which it will be a tangent, or in which it will not cut the circle.

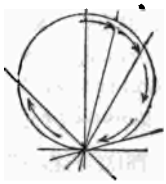


Fig. 39_6.

Cor. 6. The angle formed by a tangent and secant is (178) measured by half the included arc.

For the $\angle$ BAT, = ∟, is measured by $\frac{1}{2}$ arc BXA, (*fig.* 39_5.)
and $\angle$ BAX is measured by $\frac{1}{2}$ arc BX;
$\therefore$ [−] $\angle$ XAT is measured by $\frac{1}{2}$ arc XA.

PROPOSITION III.

The angle formed by the intersection of two secant (179) *lines is measured by the half sum or the half difference of the included arcs, according as the point of intersection is within or without the circle.*

First, let the lines AY, BZ, intersect in X, a point within the cir-

cle ABYZY′; join BY, then the measure of $\angle$ BXA = the meas. of (BYX + YBX)

= the meas. of BYX + meas. of YBX,

$= \frac{1}{2}$ arc AB + $\frac{1}{2}$ arc ZY,

$= \frac{1}{2}$ (arc AB + arc ZY).

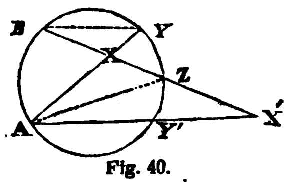


Fig. 40.

Now let the point X glide along the line BXZX′ until it take up a position X′ without the circle; as a consequence of this motion of X, the arc ZY will diminish, vanish, and finally reappear measured in the opposite direction from Z, as ZY′. But this diminution of the arc ZY may be regarded as produced by subtracting from ZY, a quantity greater than itself; therefore, the remainder ZY′ must be a minus quantity, ZY being plus:

Thus $\quad ZY' + ZY = YY', \therefore ZY' = YY' - ZY,$

or $\quad -ZY' = ZY - YY'.$

Whence the measure of $\angle$ BX′A $= \frac{1}{2}$ (arc AB − arc ZY′).

Q. E. D.

Remark. The second part of the demonstration may be made independently by joining AZ; for the measure of $\angle$ BX′A = meas. of (BZA − X′AZ) $= \frac{1}{2}$ (AB − ZY′); but it is important to arrive at a *principle* by the aid of which we may be enabled, as above, to comprehend the mutual dependencies of all the particular cases of a general proposition.

PRINCIPLE.—Whenever a Geometrical Magnitude and its Algebraical Representation can be made, by CONTINUOUS Diminution, to pass through the value nothing, and, by the same "Law of Continuity," to re-appear—then will the Geometrical Magnitude be opposite in position, and its Algebraical Representation be affected by the contrary sign. (180)

For it is evident that the magnitude, whether line, surface, solid or angle, on passing through zero, will change direction, and it is equally obvious, from the theory of algebra, that any algebraical expression, as $a - b$, by which it may be represented, will at the same time change from plus to minus or *vice versâ.*

Scholium. Hence magnitudes, as two lines measured in opposite directions, are usually affected by contrary signs, but not always. Thus, two radii of the same circle, though measured from the same point, the centre, in opposite directions, constituting a diameter, are not to be regarded the one as + and the other as −; since the one cannot be derived from the other by passing through 0.

Cor. Parallel lines intercept equal arcs; and, conversely, (181)
lines intercepting equal arcs are parallel.

For the lines AY′ and BZ being parallel, the angle (*fig.* 40.)
$X' = 0$, $\therefore$ meas. of $X' = 0$, or $\frac{1}{2}$ (arc AB − arc ZY′) $= 0$, $\therefore$ arc AB = ZY′.

Again, if AB = ZY′ then $\angle X' = \frac{1}{2}$ (arc AB − arc ZY′) $= 0$, and $\therefore$ AY′ is parallel to BZ. This corollary might have been placed under the preceding proposition. How? Join AZ.

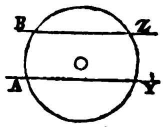


Fig. 40a.

PROPOSITION IV.

When two intersecting lines cut a circle, the product of (182)
the segments of the one, included between the point of intersection and the circumference, is equal to the product of the corresponding segments of the other.

Let the chords AY, BZ, intersect in X; join BY, AZ, (*fig.* 40.)
then the triangles AXZ, BXY, are equiangular and similar, whence the proportion

$$\frac{AX}{XZ} = \frac{BX}{XY}; \therefore AX \cdot XY = BX \cdot XZ,$$

and the theorem is proved when the point X of intersection falls within the circle. Now let the point X glide along the line BXZX′ till it takes up a position X′ without the circle; then will the segments XY, XZ, decrease, pass the value nothing, and reappear as X′Y′, X′Z, measured in the contrary direction;

$\therefore$ (180), $AX \cdot XY = BX \cdot XZ$,
becomes $AX' \cdot -X'Y' = BX' \cdot -X'Z$,
or $AX' \cdot X'Y' = BX' \cdot X'Z$. Q. E. D.

Cor. 1. Through three points not in the same straight (183)
line, a circumference of a circle may be made to pass, and but one.

Let A, B, Y, be three points not in the same straight (*fig.* 40.)
line; join BY, YA, then draw BX′ at pleasure, and make the angle YAZ = YBZ; it follows, from what has been proved, that Z will be a point of the circumference. In the same way any other point may be found; therefore the position of the three points A, B, Y, determines the positions of all the points of the circumference.

Cor. 2. A straight line cannot cut the circumference of a (184)

circle in more points than two; since through three points of a straight line no circumference can be made to pass.

Cor. 3. If from any point without a circle a tangent and (185)
secant line be drawn, the portion of the tangent intercepted between that point and the point of tangency will be a mean proportional between the segments of the secant line.

For, let the points A, Y′ approach each other so as to (*fig.* 40.)
become united in ${}_{A}T_{Y'}$; then will the chord AY′ vanish, and X′Y′A become a tangent $X'\,{}_{A}T_{Y'}$; further X′Y′ will $= X'A = X'\,{}_{A}T_{Y'}$;

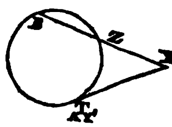


Fig. 40$_3$.

$\therefore\ X'B \cdot X'Z = X'A \cdot X'Y' = X'{}_{A}T_{Y'} \cdot X\,{}_{A}T_{Y'}$,
or $X'B : X'{}_{A}T_{Y'} :: X'\,{}_{A}T_{Y'} : X'Z$.

Cor. 4. From the same point two equal tangents can be (186)
drawn to a circle.

[Unite B, Z.]

Cor. 5. If a line be drawn bisecting the vertical angle (187)
(a, b) of a triangle, and terminating in the base, the product of the oblique sides a, b, will be equal to the product of the segments m, n, of the base, increased by the square of the bisecting line c.

Suppose a circle described about the triangle, produce the bisecting line c, so as to form the chord $c + x$, and join the extremities of $c + x$ and b; then, by similar triangles, we have

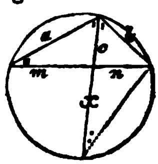


Fig. 40$_4$.

$$\frac{a}{c} = \frac{c + x}{b},$$

$$\therefore \qquad ab = c^2 + cx = mn + c^2.$$

PROPOSITION V.

The continued product of the three sides of any triangle (188)
is equal to its double area multiplied into the diameter of the circumscribing circle.

Let ABC be any triangle; suppose a circle circumscribed, and draw the diameter BD, also CP perpendicular to AB; then, by similar triangles,

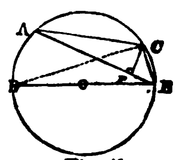


Fig. 41.

$$\frac{CP}{AC} = \frac{CB}{BD}, \ \therefore\ CP = \frac{AC \cdot CB}{BD};$$

$$\therefore\ \ 2\,\triangle\,ABC = CP \cdot AB = \frac{AB \cdot AC \cdot CB}{BD},$$

or $\quad AB \cdot BC \cdot CA = 2\,\triangle\,ABC \times BD.\quad$ Q. E. D.

PROPOSITION VI.

To find the Equation of the Circle referred to Rectangular Coördinates.

Let OX, OY, be two lines, intersecting each other at right angles in O; also let o, situated at the distance b from OX and a from OY, be the centre of any circle; further, let P be any point of the circumference, distant from OX and OY by the variable lines y and x; then will the radius, r, be the hypothenuse of a right angled triangle, of which the base will be $x-a$, and the perpendicular $y-b$, whence

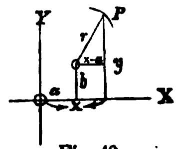


Fig. 42.

The Equation of the Circle

is
$$(y-b)^2+(x-a)^2=r^2. \tag{189}$$

Scholium. The variables x, y, are called *Coördinates* of the point, P, or of the circumference, when spoken of together; when one is to be distinguished from the other, y is denominated the *ordinate* and x the *abscissa*; OX, OY, are called the axes of coördinates, OX is the axis of x, OY that of y—O is their *origin*.

If the centre o of the circle be on OX, the axis of x, $b=0$, and the equation (189) becomes

$$y^2+(x-a)^2=r^2. \tag{190}$$

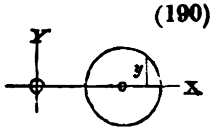


Fig. 42₂.

If, in addition to $b=0$, we make $a=0$, the centre, o, will be transported to the origin, O, and the equation becomes

$$y^2+x^2=r^2. \tag{191}$$

Resolving (191) in reference to y, we find two equal values of y for every value of x,

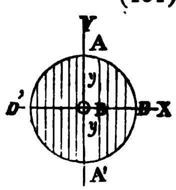


Fig. 42₃.

$$y=\pm(r^2-x^2)^{\frac{1}{2}},$$

i. e. $y=+(r^2-x^2)^{\frac{1}{2}}$, or $y=-(r^2-x^2)^{\frac{1}{2}}$.

Now, if we diminish the arc AD, its ordinate $AB=y$ diminishes also, and becomes $=0$ when $AD=0$; finally AD reappearing as A′D measured in the opposite direction, its ordinate A′B reappears, likewise measured in the opposite direction from B, and

is therefore $= -y$, according to the principle laid down in (180). Whence,

Cor. 1. The circle is perfectly symmetrical, any diameter, (192)
as DD′, bisecting all the chords to which it is perpendicular [*fig.* 42_2].

Cor. 2. The perpendicular which bisects a chord passes (193)
through the centre of the circle and bisects the arc.

Cor. 3. The greater chord is less distant from the centre, (194)
since the semichord y increases as x diminishes; and the diameter is, therefore, the greatest straight line that can be drawn in a circle.

By comparing equations (189), (190), (191), with the figures (42), (42_2), (42_3), which illustrate them, it is obvious that the coordinates y, x, represent, in general, totally different quantities in these different forms, so that one cannot be combined with another, as in ordinary algebraical operations. But if the circles intersect, as in figure 42_4, then the point of intersection, I, being the same for both circumferences, the coördinates of this point, y_i, x_i, will satisfy the equations of both circles, and from (191) and (190) denoting the different radii by r, r_2, we have

$$y_i^2 + x_i^2 = r^2,$$

$$y^2 + (x_i - a)^2 = r_2^2;$$

whence $$2ax_i - a^2 = r^2 - r_2^2.$$

Resolving the last equation in reference to a, we find

$$a = x_i \pm (x_i^2 + r_2^2 - r^2)^{\frac{1}{2}};$$

whence, for any compatible values of x_i, r_2, r, we find two values for the distance of the centres, a; or, for any given radii and a given chord, the circles can be made to intersect in two ways.

If $r_2 < r$, the values of a will be both $+$, the one greater than x_i, the other less, and the centres will be

on opposite sides of the ordinate, y_i,

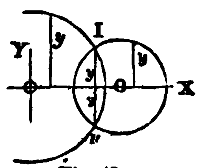


Fig. 42_4.

or on the same side, accordingly.

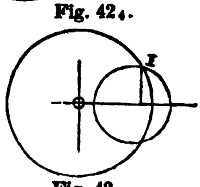


Fig. 42_5.

If $r_2 > r$, the values of a will be, $\left\{\begin{array}{l}\text{one plus,}\\ \\ \text{the other}\\ \text{minus;}\end{array}\right\}\left\{\begin{array}{l}\text{as rep. in}\\ \\ \\ \text{and}\end{array}\right.$

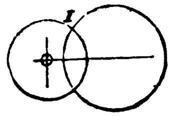


Fig. 42$_6$.

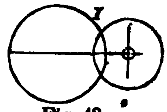


Fig. 42$_7$.

If we take $-x_i$ instead of $+x_i$, $a = x_i \pm (x_i^2+r_2^2-r^2)^{\frac{1}{2}}$, becomes $a = -x_i \pm (x_i^2+r_2^2-r^2)^{\frac{1}{2}}$, or $-a = x_i \mp (x_i^2+r_2^2-r^2)^{\frac{1}{2}}$; whence we have the same constructions over again, only on the opposite side of the origin.

If we substitute the value of $x = \dfrac{a^2+r^2-r_2^2}{2a}$, in $y_i^2 + x_i^2 = r^2$,

we find $y_i = \pm\left[r^2 - \left(\dfrac{a^2+r^2-r_2^2}{2a}\right)^2\right]^{\frac{1}{2}}$;

whence it follows that the circumferences intersect in two points, I, I′, equally distant from the middle, B, of their common chord; and ∴ that, as long as y_i has a real value, the triangle OI$_0$ must be possible. From what has been demonstrated, we have the following corollaries:

Cor. 4. Two circles of given radii will have for a common chord four positions of intersection. (195)

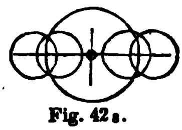


Fig. 42$_8$.

Cor. 5. The line joining the centres of intersecting circles, is perpendicular to, and bisects their common chords. (196)

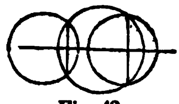


Fig. 42$_9$.

Cor. 6. In order that two circles intersect, of the three quantities, their radii and the distance of their centres, any two must be greater than the third. (197)

Cor. 7. The distance of the centres of two tangent cir- (198)

cles, is equal to the sum or difference of their radii, according as they touch externally or internally.

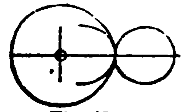


Fig. 42_{10}.

Cor. 8. The line joining the centres of tangent circles, passes through the point of tangency, and the converse. (199) [*fig*. 42_{10}.]

EXERCISES.

1°. What is the greatest triangle that can be inscribed in a semicircle? *Ans.* An isosceles triangle, standing upon the diameter.

2°. What is the greatest rectangle that can be inscribed in a given circle? *Ans.* A square.

3°. What is the maximum triangle that can be inscribed in a given circle and standing on its radius?

Ans. A triangle right angled at the centre.

4°. How many boards 15 inches wide and an inch thick can be cut from a log 20 inches in diameter? *Ans.* $5\sqrt{7}$.

5°. What must be the diameter of a water wheel to fit an apron $2a$ feet across and b feet deep?

6°. How much must a plank be cut out to make a felly 1 ft. 6 in. long to a wheel 6 ft. in diameter, the measures being taken on the inside?

7°. What is the radius of the largest circle that can be cut from a triangular plate of silver, measuring 2·3, 3, 3·5 inches on its sides?

8°. Three brothers, residing at the several distances of 10, 11, 12 chains from each other, are to dig a well which shall be equally distant from them all. What must be that distance?

9°. The diameters of the fore and hind wheels of a carriage are 4 and 5 feet, and the distance of their centres 6 feet. At what point will a line joining these centres intersect the ground, supposed to be a plane?

10°. There are two wheels situated in the same vertical plane, and their centres in the same vertical line; the largest, the centre of which is 10 feet below the floor, is 8 feet in diameter, and the smaller, the centre of which is 6 feet above the floor, is one foot in diameter. Where must we cut through the floor for the passage of the strap that is to embrace the wheels?

11°. The same as in the above, except that the strap is to cross between the wheels.

12°. The same conditions as in 10°, except that the centre of the lower wheel is 4 feet from the plumb line dropped from the centre of the wheel above the floor.

13°. It is required to construct three equal friction wheels to run tangent to each other and to an axle two inches in diameter. What must be their common radius, and what the radius of the circular bed cut for them in the centre of a wheel?

14°. If the top-masts of two ships, having elevations of 90 and 100 feet above the level of the sea, are seen from each other at the distance of 25·7 miles, what is the diameter of the earth?

15°. How far can the Peak of Teneriffe be seen at sea?

16°. How far will a water level fall away from a horizontal line, sighted at one end in a distance of one mile, the diameter of the earth being estimated at 7,960 miles?

17°.* If AC, one of the sides of an equilateral triangle ABC, be produced to E, so that CE shall be equal to AC; and if EB be drawn and produced till it meets in D, a line drawn from A at right angles to AC; then DB will be equal to the radius of the circle described about the triangle.

18°. If an angle B of any triangle ABC, be bisected by the straight line BD, which also cuts the side AC in D, and if from the centre A with the radius AD, a circle be described, cutting BD or BD produced in E; then BE : BD : : AD : CD.

19°. Let ABC be a triangle right angled at B; from A draw AD parallel to BC, and meeting in D, a line drawn from B at right angles to AC; about the triangle ADC describe a circle, and let E be the point in which its circumference cuts the line AB or AB produced; then AD, AB, BC, AE, are in continued proportion.

20°. Let ABC be a circle, whose diameter is AB; and from D any point in AB produced, draw DC touching the circle in C, and DEF any line cutting it in E and F; again, draw from C a perpendicular to AB, cutting EF in H; then,

$$ED^2 : CD^2 : : EH : FH.$$

21°. Let ABC be a circle, and from D, a point without it, let three straight lines be drawn in the following manner: DA touching the circle in A, DBC cutting it in B and C, and DEF cutting it in E and F; bisect the chord BC in H, draw AH, and produce it till it meets the circumference in K; draw also KE and KF cutting BC in G and L. The lines HG and HL are equal.

* "Prize Problems," Yale College, 1840

SECTION SECOND.

The Ellipse.

Def. 1. An *ellipse* is a plane curve described by the intersection of two radii, varying in such manner as to preserve in sum the same constant quantity, while they revolve about two fixed points as centres.

PROPOSITION I.

To find the Equation of the Ellipse.

Let P be any point of the curve, formed by the intersection of the variable radii $a + u$, $a - u$, which are equal in sum to a constant quantity, $2a$, and which revolve about two fixed points, F, F′, distant from each other by the line $2c$. Take the middle point O between F, F′, for the origin of rectangular coördinates, the line passing through F, F′, being the axis of X. There results,

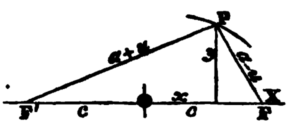


Fig. 43.

$$(a+u)^2 = y^2 + (c+x)^2$$

and $$(a-u)^2 = y^2 + (c-x)^2;$$

$\therefore$ [+], $$a^2 + u^2 = y^2 + c^2 + x^2,$$

and, [−], $$au = cx,$$

or $$u = \frac{c}{a} \cdot x;$$

$\therefore$ $$a^2 + \frac{c^2}{a^2} \cdot x^2 = a^2 + u^2 = y^2 + c^2 + x^2,$$

or $$a^4 + c^2x^2 = a^2y^2 + a^2c^2 + a^2x^2;$$

$\therefore$ $$a^2y^2 + (a^2 - c^2)x^2 = a^2(a^2 - c^2), \qquad (200)$$

the equation of the ellipse.

Now, it is obvious that, if the curve cuts the axis of x, y for that point will be reduced to nothing; therefore, if we make $y = 0$, and denote by $x_{y=0}$ what x becomes for this value of y, we have (200)

$$a^2 \cdot 0^2 + (a^2 - c^2)x_{y=0}^2 = a^2(a^2 - c^2);$$

$\therefore$ $$x_{y=0} = \pm a; \text{ hence}$$

Cor. 1. The ellipse cuts the axis of abscissas at equal (201) distances on the right and left of the origin, which distance $= a$.

Fig. 43₂.

When $x = 0$, the curve cuts the axis of y, but this condition gives (200)

$$a^2y^2_{x=0} + (a^2 - c^2)\cdot 0^2 = a^2(a^2 - c^2);$$

$$\therefore \qquad y_{x=0} = \pm(a^2 - c^2)^{\frac{1}{2}}; \text{ hence,}$$

Cor. 2. The ellipse cuts the axis of ordinates at equal (202) distances above and below the origin, which distance, denoted by

$$b, = (a^2 - c^2)^{\frac{1}{2}}.$$

Fig. 43₃.

Def. 2. The line MON $= 2a$, is denominated the *Major Axis* or the *Conjugate Diameter*, and passes through the Foci,* F, F′; the line POQ $= 2b$, is the *Minor Axis* or the *Transverse Diameter*, being perpendicular to the former. [b can never > than a.]

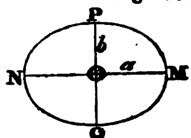


Fig. 43₄.

If in (200) we substitute b^2 for $(a^2 - c^2)$, the equation of the ellipse becomes

$$a^2y^2 + b^2x^2 = a^2b^2, \text{ or } \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1, \qquad (203)$$

where the constants are the semimajor and semiminor axes.

Cor. 3. The ellipse is symmetrical in reference to both (204) axes; since,

For every value of x, whether + or −,

we have $y = \pm\left(b^2 - \frac{b^2}{a^2}\cdot x^2\right)^{\frac{1}{2}}$, two equal values of y, one +, the other −;

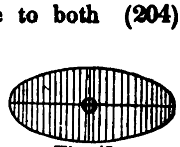


Fig. 43₅.

and, for every value of y, whether $+y$ or $-y$,

we have $x = \pm\left(a^2 - \frac{a^2}{b^2}\cdot y^2\right)^{\frac{1}{2}}$, two equal values of x, one +, the other −; hence,

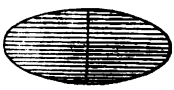


Fig. 43₆.

Cor. 4. The major axis bisects all chords parallel to the (205) minor, and the minor axis bisects all chords parallel to the major.

Cor. 5. The origin bisects all chords drawn through it, (206)

* Foci, plural of focus, fire-place.

and is, consequently, the *Centre* of the Ellipse; therefore these chords are *Diameters*.

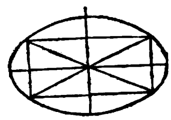


Fig. 43₇.

Cor. 6. Diameters, equally inclined to the major axis, are equal; and the converse. (207)

Cor. 7. Any ordinate of the ellipse is to the corresponding ordinate of the circle, described on the major axis, as the semiminor axis is to the semimajor. (208)

For let y, Y, be corresponding ordinates of the ellipse and circle; we have (203)

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2),$$

and (?) $Y^2 = a^2 - x^2$;

$\therefore \quad \frac{y^2}{Y^2} = \frac{b^2}{a^2}, \therefore y : Y :: b : a; \therefore$

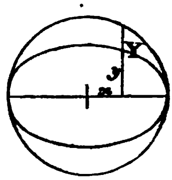


Fig. 43₈.

Cor. 8. The circle, described on the major axis, circumscribes the ellipse. Hence, (209)

Cor. 9. The angle embraced by chords, drawn from any point of the ellipse to the extremities of the major axis, is obtuse. (210)

[How?]

Fig. 43₉.

Cor. 10. Any abscissa of the ellipse is to the corresponding abscissa of the circle described on the minor axis, as the semimajor axis to the semiminor. (211)

For, we have

$$x^2 = \frac{a^2}{b^2}(b^2 - y^2),$$

and $\quad X^2 = b^2 - y^2$;

$\therefore \quad x : X :: a : b.$

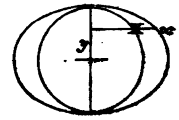


Fig. 43₁₀.

Cor. 11. The circle described on the minor axis is inscribed in the ellipse. (212)

Cor. 12. The angle embraced by chords drawn from any point of the ellipse to the extremities of the minor axis, is acute. (213)

Def. 3. The double ordinate drawn through the focus, is denominated the *Parameter* of the major axis, and sometimes the *Latus Rectum*.

To find the parameter we have only to make $x = c$ in (200); whence there results

$$a^2y^2_{x=c} + (a^2 - c^2)c^2 = a^2(a^2 - c^2),$$

$$\therefore \quad Parameter = 2y_{x=c} = 2 \cdot \frac{a^2 - c^2}{a} = \frac{2b^2}{a} = \frac{4b^2}{2a} = \frac{(2b)^2}{2a};$$

or $\qquad 2a : 2b :: 2b : Parameter;$ i. e.

Cor. 13. The *Parameter* is a third proportional to the major and minor axes. (214)

Putting the parameter $= p$, and substituting in (203) we get

$$y^2 = \frac{p}{2a}(a^2 - x^2), \tag{215}$$

for the equation of the ellipse, in terms of the parameter and semi-major axis.

We may transform (200) into

$$a^2y^2 = (a^2 - c^2)(a^2 - x^2),$$

or
$$y^2 = \left(1 - \frac{c^2}{a^2}\right)(a^2 - x^2),$$

or
$$y^2 = (1 - e^2)(a^2 - x^2), \tag{216}$$

putting
$$e = \frac{c}{a}. \tag{217}$$

Def. 4. We call e the *Eccentricity* because it expresses the ratio of the distance, c, of the focus from the centre to the semi-major axis, and thus determines the form of the ellipse, as round or flat. When the eccentricity is $= 0$, the ellipse becomes a circle. Equation (216) is that of the ellipse, referred to its eccentricity and semimajor axis, and is convenient in astronomy.

It is sometimes desirable to have the equation of the ellipse, when the left hand extremity of the major axis is made the origin of abscissas. In order to this, we have only to substitute $x - a$ instead of x in (203), as the new x exceeds the old by a, y remaining the same; which done, there results,

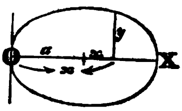


Fig. 43II.

$$y^2 = \frac{b^2}{a^2}(2ax - x^2). \tag{218}$$

The origin might be transported to any other point, either in the curve or elsewhere, by changing the value of y as well as that of x.

Scholium. It is easy to show that any equation referred to rectangular coördinates, and of the form

$$py^2 + qx^2 = r,$$

is the equation of an ellipse; for we have

$$\frac{y^2}{\frac{r}{p}} + \frac{x^2}{\frac{r}{q}} = 1,$$

which will agree with (203), by putting

$$b^2 = \frac{r}{p},\ a^2 = \frac{r}{q},$$

or by finding $b = \sqrt{\left(\frac{r}{p}\right)},\ a = \sqrt{\left(\frac{r}{q}\right)}$.

PROPOSITION II.

A tangent to the Ellipse makes equal angles with the lines joining the foci and the point of tangency. (219)

Let P be any point of the ellipse, through which the line P′PR is drawn so as to make the angles FPR, F′PP′, equal; then will P′PR be tangent at the point P. For, producing F′P to Q, and making PQ = PF, we have

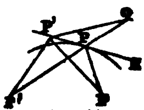


Fig. 44.

$$F'P' + P'Q > F'P + PQ = F'P + PF.$$

Now, if P′PR be not a tangent, let the second point, in which it cuts the curve, be P′, which we are at liberty to suppose, since P′ may be any point of P′PR; then the definition of the ellipse gives

$$F'P' + P'F = F'P + PF, \text{ which is less than } F'P' + P'Q;$$

$$\therefore P'F < P'Q, \text{ and } \therefore \angle P'PF < P'PQ, \text{ or } FPR > QPR = F'PP',$$

which is contrary to the hypothesis; hence, so long as the $\angle$ FPR = F′PP′, the line P′PR cannot cut the ellipse, and is tangent to it. Q. E. D.

Scholium. It is on this account that F, F′, are called the *Foci* of the ellipse; since, from the principle of light and heat, that the angle of reflection is equal to the angle of incidence, if the curve were a polished metallic hoop, and a flame placed at F, the rays, reflected from all points of the ellipse, would pass through F′.

Cor. 1. The tangents drawn through opposite extremities of any diameter, are parallel. (220)

[See (206), (99).]

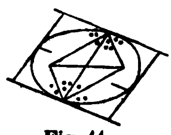


Fig. 44a.

Cor. 2. A parallelogram circumscribing an ellipse will be formed by drawing tangents through the opposite extremities of any two diameters. (221)

Cor. 3. The tangents drawn through the extremities of the axes are at right angles to them, and the circumscribing figure becomes a rectangle. (222)

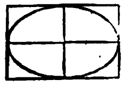


Fig. 44_3.

Cor. 4. The *Normal*, or line drawn through the point of tangency perpendicular to the tangent, bisects the angle embraced by the lines drawn from the point of tangency to the foci. (223)

Fig. 44_4.

Cor. 5. The axes are normal to the tangents drawn through their extremities. (224)

PROPOSITION III.

To find where the normal intersects the axis of abscissas.

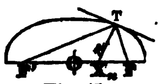


Fig. 45.

Let $TX_n = n$ be the normal, intersecting the axis of X in X_n; from (142) we have the proportion

$\frac{X_nF}{X_nF'} = \frac{TF}{TF'}$, or $\frac{c - x_n}{c + x_n} = \frac{a - u}{a + u}$, putting $OX_n = x_n$,

$\therefore$ (40, 3°), $\frac{2x_n}{2c} = \frac{2u}{2a}$, $\therefore x_n = \frac{c}{a} \cdot u$, in which, substituting the value of u, $= \frac{c}{a} \cdot x$, found under Prop. I., we have

$$OX_n = x_n = \frac{c^2}{a^2} \cdot x, \text{ the point required.} \quad (225)$$

We observe that when c becomes $= 0$, $x_n \left[= \frac{c^2}{a^2} \cdot x\right]$ becomes $= 0$, and $\therefore$ the normal passes through the centre, O, as it ought to do, since the ellipse becomes then a circle described with the radius a. We have

$$\textit{Subnormal} = XX_n = OX - OX_n = x - x_n \quad (226)$$
$$= x - \frac{c^2}{a^2} \cdot x = \frac{a^2 - c^2}{a^2} \cdot x = \frac{b^2}{a^2} \cdot x.$$

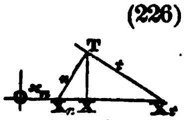


Fig. 45_2.

We have (129),

$$\frac{\text{Subtangent } [=XX_t]}{\text{Ordinate } [TX]} = \frac{\text{Ordinate } [TX]}{\text{Subnormal } [XX_n]};$$

$$\therefore \quad \textit{Subtangent} = \frac{(\text{Ordinate})^2}{\text{Subnormal}} = \frac{TX^2}{XX_n} \qquad (227)$$

$$= \frac{y^2}{\frac{b^2}{a^2}\cdot x} = \frac{\frac{b^2}{a^2}(a^2-x^2)}{\frac{b^2}{a^2}\cdot x} = \frac{a^2-x^2}{x}.$$

We observe here that the subtangent is independent of the value of b; therefore,

Cor. If upon the same major axis, any number of ellipses (228) of different breadths, and also a circle be described, then their tangents, drawn to the same abscissa will intersect the axis of x in the same point.

Fig. 45a.

EXERCISES.

1°. Prove that if two points of a straight line glide along two other lines intersecting at right angles, any third point of the first line will describe an ellipse.

2°. The equation of an ellipse referred to rectangular coördinates is

$$9y^2 + 4x^2 = 36.$$

Required the distance from the origin to the point in which the normal cuts the axis of x, the abscissa of the point of tangency being $x_t = 1$.

3°. Where does the normal in 2° cut the axis of y?

4°. Required the subtangent in 2°.

5°. Required the length of tangent in 2° intercepted by the axes of coördinates.

6°. How far distant from the centre are the foci in 2°?

7°. What is the eccentricity in 2°?

8°. Required the parameter in 2°.

9°. It is required in 2° to transfer the origin to a point in the ellipse, the abscissa of which shall be $x = 1{\cdot}2$, the new axes being parallel to the old.

This will be done by substituting for y and x, $y + y_1$ and $x + x_1 = x + 1{\cdot}2$, and observing that

$$9y_1^2 + 4x_1^2 = 36, \text{ or } 9y_1^2 + 4\cdot 1{\cdot}2^2 = 36.$$

10°. Given $9y^2 - 90y + 4x^2 + 56x + 385 = 0$, the equation of a

curve referred to rectangular coördinates; it is required to ascertain whether the curve be an ellipse.

Substitute for y, $y+y_1$ and for x, $x+x_1$; find the coëfficients of the first power of y and the first power of x in the new equation, and put these coëfficients separately $=0$, from which deduce the values of y_1, x_1. The resulting equation will be found to be $9y^2+4x^2=36$.

11°. According to Sir John F. W. Herschel, the equatorial diameter of the earth is 7925·648 miles, and its polar diameter 7899·170 miles. The situation of a place in north latitude is such that a perpendicular dropped upon the earth's axis will intersect it at the distance of 2456 miles from the centre. Required the point in which the direction of a plumb line suspended at the place, will cut the axis of the earth; the meridian being regarded as an ellipse and the plumb as perpendicular to the surface of still water.

SECTION THIRD.

The Hyperbola.

Definition. A *Hyperbola* is a plane curve described by the intersection of two radii varying in such manner as to preserve in difference the same constant quantity, while they revolve about two fixed points as centres.

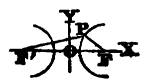


Fig. 46.

Ordering all things as for the equation of the ellipse, except that the radii are to be denoted by $u+a$, $u-a$, we find

$$a^2y^2-b^2x^2=-a^2b^2, \tag{229}$$

for the equation of the hyperbola.

The properties of the hyperbola are obviously analogous to those of the ellipse. The student will exercise himself in ascertaining them.

It has been remarked that the equation of the ellipse, $a^2y^2+b^2x^2=a^2b^2$, becomes that of the circle, $y^2+x^2=a^2$, by making $b=a$; so the equation of the hyperbola becomes $y^2-x^2=-a^2$, by putting $b=a$, an equation much resembling that of the circle; hence, the curve which it represents, the *Equilateral Hyperbola*, possesses properties analogous to those of the circle.

SECTION FOURTH.

The Parabola.

Definition. The *Parabola* is a plane curve, such that any one of its points is equally distant from a fixed point and a line given in position, which line is denominated the *Directrix.*

PROPOSITION I.

To find the Equation of the Parabola.

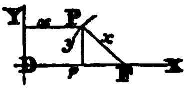


Fig. 47.

Take the directrix, D, for the axis of y, and for the axis of x the perpendicular to D, drawn through the fixed point, F, whose distance from D we will indicate by p. Then, by the definition, P representing any point of the curve, we have

$$y^2 + (p - x)^2 = x^2,$$

$$\therefore \quad y^2 = 2px - p^2, \qquad (230)$$

is an equation of the parabola.

If in this equation we make $y = 0$ for the purpose of finding where the curve cuts the axis of x, there results

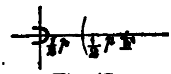


Fig. 47₂.

$$x_{y=0} = \tfrac{1}{2}p;$$

$\therefore$ *Cor.* 1. The Parabola cuts the axis of x midway between the fixed point F and the directrix. (231)

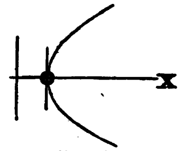


Fig. 47₃.

In order to transport the origin to this point, we have only to substitute in (230) for x, $x + \frac{1}{2}p$; doing which, there results

$$y^2 = 2px, \qquad (232)$$

the equation of the parabola.

Here we observe that as x increases y increases, and that without limit, and for every value of x there results two equal values of y; also, x does not admit of any minus value, since in that case y, $[= (-x)^{\frac{1}{2}}]$, would be imaginary; hence,

Cor. 2. The parabola opens indefinitely to the right in two symmetrical branches, but, unlike the Hyperbola, has no branch on the left of the origin. (233)

PROPOSITION II.

To draw a tangent to the Parabola.

Let PP_2 be any curve, cut by the line P_2PX_s, intersecting the axis of x in X_s; then, denoting the coördinates of the point P by y, x, and of P_2 by $y+k$, $x+h$, we have

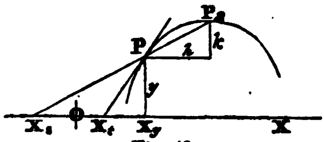


Fig. 48.

$$\frac{y}{\text{subsecant } X_sX_y} = \frac{k}{h}. \tag{234}$$

Now it is obvious, that, if the point P_2 be made to approach the point P until the two coincide, the line P_2PX_s will cease to be a secant, and consequently become a tangent at the point P. To effect this we have only to make h diminish till $h=0$, and $\therefore$ $k=0$; indicating what the ratio $\frac{k}{h}$ becomes under this condition, by including $\frac{k}{h}$ in brackets, we have

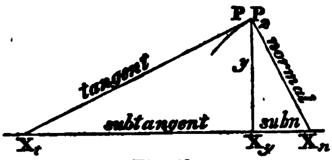


Fig. 48₂.

$$\frac{y}{\text{subtangent } X_tX_y} = \left[\frac{k}{h}\right], \tag{234$_2$}$$

where the value $\left[\frac{k}{h}\right]$ is to be drawn from the equation of the curve.

Applying this process to the parabola, we have

$$y^2 = 2px - p^2,$$

and $$(y+k)^2 = 2p(x+h) - p^2;$$

$\therefore$ $$(y+k)^2 - y^2 = 2p(x+h) - 2px,$$

or $$2yk + k^2 = 2ph;$$

$\therefore$ $$\frac{k}{h} = \frac{2p}{2y+k}, \quad \therefore \quad \left[\frac{k}{h}\right] = \frac{p}{y},$$

which substituted in (234₂) gives

$$\frac{y}{\text{subtangent}} = \frac{p}{y};$$

$$\therefore \qquad \text{subtangent} = \frac{y^2}{p}. \tag{235}$$

But we have in all cases [*fig*. 48₁.]

$$\frac{\text{subnormal } X_yX_n}{y} = \frac{y}{\text{subtangent}}, \tag{236}$$

$\therefore$ for the parabola we find

$$\text{subnormal } X_yX_n = \frac{y^2}{\text{subt.}} = \frac{y^2}{\frac{y^2}{p}} = p. \tag{237}$$

Whence we have

$FX_n = X_yX_n - X_yF = p - (p - x) = x$;

but $FP = x$, $\therefore$ $FX_n = FP$,

$\therefore$ $\angle FX_nP = FPX_n$;

therefore, if we draw a line PX_2 parallel to the axis of x, the angle $X_nPX_2 = X_nPF$. It follows that all rays of light or heat, or of sound, parallel to the axis of the parabola, will be collected in F. Hence F is denominated the *Focus* of the parabola.

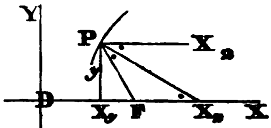


Fig. 48₁.

Cor. 1. The points where the normal intersects the parabola and its axis, are equally distant from the focus, and the normal is consequently equally inclined to the axis and to the radius vector, terminating in the same point of the curve; $\therefore$ (238)

Cor. 2. The tangent makes equal angles with the axis and the line joining the point of tangency and the focus. (239)

In equation (232) the abscissa of the focus is $\frac{1}{2}p$;

$$\therefore \qquad Parameter = 2y_{x=\frac{1}{2}p} = 2p. \tag{240}$$

Scholium I. The method here employed for drawing a tangent to the parabola is obviously applicable to all curves; and it is recommended to the student to make himself familiar with it by drawing tangents to the circle, ellipse and hyperbola, and to verify his results by the properties already demonstrated in regard to these curves.

Scholium II. We must not pass unnoticed the remarkable symbol, by which we have readily arrived at important relations. Since in $\left[\frac{k}{h}\right]$ we have reduced both h and k to zero, it is natural to regard this expression as equivalent to $\frac{0}{0}$, and this in itself, abstractly considered, has no meaning at all, for to it we cannot attach any idea

independent of its origin. But to regard the symbol $\left[\frac{k}{h}\right]$ as destitute of signification, or not indeed as possessing an important one, would be to attribute to it an altogether erroneous interpretation. In truth, it not only indicates a quantity, but that quantity as evolved, by a peculiar operation, from specific conditions.

The symbol $\left[\frac{k}{h}\right]$ signifies,

1°. There are two quantities which are regarded as variable, y and x.

2°. y is regarded as depending upon x.

3°. *Increments* [increases], k and h, are given to these variables, y, x.

4°. The ratio, $\frac{k}{h}$, of the increment of y to that of x is found.

5°. That particular value, $\left[\frac{k}{h}\right]$, of this ratio is taken, which is obtained by diminishing h, and consequently k, to zero.

It is also to be observed that the ratio, $\left[\frac{k}{h}\right]$, will generally itself be a variable quantity. Indeed, in this particular case of a tangent to the parabola, we have $\left[\frac{k}{h}\right] = \frac{p}{y}$, which may vary from $\frac{p}{y=0}$ = infinity to $\frac{p}{y = \text{infinity}} = 0$. In the next book we shall give the symbol, $\left[\frac{k}{h}\right]$, a name, and a further investigation.

PROPOSITION III.

If a curve be such that the distance of any point of it to a point fixed in space shall bear a constant ratio, e, *to the distance of the same point of the curve to a given line or Directrix, then will the curve be either the Ellipse, Hyperbola or Parabola, according as the ratio,* e, *is less than, greater than, or equal to unity,* [$e \lesseqgtr 1$]. (241)

Let the fixed line or directrix be the axis of y, and the perpendicular to it drawn through the fixed point F, the axis of x, and let the distance of F from the origin be denoted by d; there results,

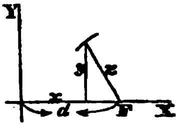


Fig. 49.

$$y^2+(d-x)^2=z^2=e^2x^2, \text{ since } \frac{z}{x}=e, \text{ by hypothesis};$$

$$\therefore \quad y^2+(1-e^2)x^2-2dx+d^2=0, \text{ which becomes at once}$$
$$y^2=2dx-d^2,$$

the equation of the parabola, when $e=1$, or $(1-e^2)=0$. In order to make the term $2dx$, affected by the first power of x, disappear, we will transport the origin to the right a distance $=m$, so that we shall have $x+m$ instead of x,

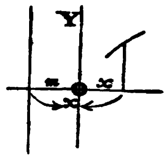


Fig. 49a.

$$\therefore \quad y^2+(1-e^2)(x+m)^2-2d(x+m)+d^2=0,$$

or $$y^2+(1-e^2)x^2+[(1-e^2)\cdot 2m-2d]x+(1-e^2)m^2-2dm+d^2=0,$$

from which, attributing such a value to m as to make the term affected by the first power of x disappear, we have

$$(1-e^2)\cdot 2m-2d=0,$$

and $$y^2+(1-e^2)x^2+(1-e^2)m^2-2dm+d^2=0;$$

whence, eliminating m, there results

$$y^2+(1-e^2)x^2=\frac{e^2d^2}{1-e^2},$$

or $$\frac{y^2}{\frac{e^2d^2}{1-e^2}}+\frac{x^2}{\frac{e^2d^2}{(1-e^2)^2}}=1,$$

which is (1°) the equation of an ellipse $\frac{y^2}{b^2}+\frac{x^2}{a^2}=1$,

or (2°) the equation of a hyperbola, $\frac{y^2}{-b^2}+\frac{x^2}{a^2}=1$,
according as $1-e^2$ is $+$, or $-$, that is, $e<1$, or $e>1$. Q. E. D.

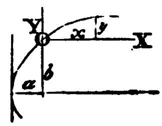
Scholium. It is to be observed that the Ellipse, Circle, Hyperbola, and Parabola may be represented by the same general equation, and are therefore to be regarded as nothing more than species of the same curve.

EXERCISES.

1°. From the top of a tower 48 feet high, a cannon ball is fired in a horizontal direction with a velocity of 1000 feet per second. Required the distance from the foot of the tower where the ball will strike the horizontal plane on which it stands; no allowance being made for atmospheric resistance, and the vertical descent being in the times 1, 2, 3, &c., seconds, $1^2 \cdot (16\frac{1}{12})$, $2^2 \cdot (16\frac{1}{12})$, $3^2 \cdot (16\frac{1}{12})$, &c., feet.

2°. To transport the origin to a point of the Parabola, the new axes being parallel to the old.

Let b, a, be the coördinates of the new origin referred to the old axes; we have



$$(y+b)^2 = 2p(x+a),$$

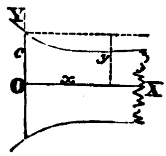
but $$b^2 = 2pa;$$

$\therefore$ $y^2 + 2by = 2px$, is the equation required.

Fig. 50.

3°. To ascertain whether a board cut from a log next the roots without having been squared, may be regarded as an inverted parabola.

Let the middle line of the board be taken for the axis of x and the broader end for the axis of y; the preceding problem gives us



$$(c-y)^2 + 2b(c-y) = 2px,$$

where there are three constants, b, c, p, to be determined, one of which, c = half the width of the broader end, may be supposed known. We must therefore determine the values of b and p from values of x and y taken in two different places, and then see if b and p remain the same, or nearly the same, for measures taken throughout the length of the board.

Fig. 51.

4°. The length of a board, of the form given in 3°, is 8 feet, the ends are 4 and 2·4 ft. broad, and the breadth of the middle is 3 ft. Required the equation of its edge, the axes of coördinates being as in the last.

Ans. $(2-y)^2 + \cdot 7(2-y) = \cdot 15x$,
or $y^2 - 4{\cdot}7y = \cdot 15x - 5{\cdot}4$.

5°. It is required to form a gauge by which to turn a parabolic mirror 18 inches in diameter, and having a focal distance of 10 inches, measuring from the diameter. Required the depth of the mirror and its equation.

Ans. Depth = 1·7268. Equation, $y^2 = 46{\cdot}9072 \cdot x$.

6°. Fold the lower corner of the left hand page, so as to make the area of the folded part constantly equal to a given square (a^2) the side of which is a; and find the *locus* of the corner, or the curve in which it is constantly situated.

If we take the position of the corner before folding for the origin of coördinates and the two edges of the leaf for the axes, we shall find

$$(y^2+x^2)^2=2(2a)^2yx,$$

for the equation sought; from which it appears that the curve is symmetrical in reference to the axes or edges of the leaf, and that it begins and ends at the origin, since $x=0$ gives $y=0$.

PART SECOND.

ANALYTICAL FUNCTIONS, PLANE TRIGONOMETRY, AND SURVEYING.

BOOK FIRST.

ANALYTICAL FUNCTIONS.

SECTION FIRST.

Primitive and Derivative Functions of the Form x^m

Definition 1. When one quantity depends upon another, so that the first varies constantly by a continuous change of value in the second, the first is said to be a FUNCTION of the second. The first is also called the *function* and the second *the independent variable*.

Thus, in the equation of the parabola, $y^2 = 2px$, y is a function of x, or y is the function and x the variable.

Def. 2. An *increasing* function is one which increases when the variable increases, and a *decreasing* function is such that it decreases when an increment is given to the independent variable. Thus, in the equation $y^2 = 2px$, y is an increasing function of x, but in the equation of the ellipse, $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$, y is a decreasing function.

Def. 3. An *explicit* function is one in which the value of the function is developed or expressed; an *implicit* function is one in which the value of the function is implied. Thus $y = (2p)^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$, is an explicit function; and y is an implicit function of x, or x an implicit function of y, in the equation $a^2y^2 + b^2x^2 = a^2b^2$.

As it is useful to denote quantities and their combinations by letters and signs, so it will be found advantageous to indicate different functions by appropriate symbols. The letters f, F; f_1, F_1, &c.; f', F', &c., as well as others derived from the Greek alphabet, such as ϕ (phi), ψ (psi), are employed for this purpose. Thus

fx signifies, not that f is multiplied into x, but that some operation is to be performed upon x, such as squaring it, taking its square root, &c., in which case we have $fx = x^2$, or $fx = \sqrt{x}$, &c.; fx will signify a different function or operation from $f_1 x$, and the distinction in reading will be, the f function of x, the f sub one function of x. Further, the small letters may be used to express known relations, while the capitals denote unknown functions.

The letter h will always be employed to denote the *Increment* or increase of the independent variable x, while k will as constantly indicate the consequent increment of the function y.

Def. 4. The *Derivative* of a function, or the *Derived Function*, or simply the DERIVATIVE, is the function obtained by taking that particular value of the ratio of the increment of the dependent variable to that of the independent when the latter reduces to zero.

Thus, if $\quad y = fx = x^2,$

then* $\quad y + k = (x+h)^2 = x^2 + 2xh + h^2;$

$\therefore \quad k = 2xh + h^2;$

$\therefore \quad \dfrac{k}{h} = 2x + h;$

and $\therefore \left[\dfrac{k}{h}\right] = 2x$, the value of the ratio $\dfrac{k}{h}$ when $h = 0$ and $\therefore k = 0$,

indicated by the brackets [ ], is the *derivative* of the function x^2. And, as y is put for the function x^2, so it is natural to indicate the derived function $2x$ by y', which will therefore be read the *derivative* of y —, and we shall write

$$y' = \left[\frac{k}{h}\right] = 2x.$$

The letter f' will be employed for the same purpose, and $f'x$ will be read the derived function of x. As a further illustration, let us take the function

$$y = fx = ax^3 + b,$$

then* $\quad y + k = f(x+h) = a(x+h)^3 + b = a(x^3 + 3x^2h + 3xh^2 + h^3) + b.$

$\therefore \quad k = f(x+h) - fx = a(3x^2h + 3xh^2 + h^3),$

and $\quad \dfrac{k}{h} = \dfrac{f(x+h) - fx}{h} = a(3x^2 + 3xh + h^2);$

* Increasing x by h, and, consequently, y by k, or changing x into $x+h$ and y into $y+k$, or, y depending on x by a constant law, $y+k$ is the *same* function of $x+h$ that y is of x.

$$\therefore \quad y' = f'x = \left[\frac{k}{h}\right] = \left[\frac{f(x+h)-fx}{h}\right] = a \cdot 3x^2.$$

It is obvious that y', $f'x$, $\left[\frac{k}{h}\right]$, $\left[\frac{f(x+h)-fx}{h}\right]$, are but different expressions for the same thing. Observe further that y', being equal to $a \cdot 3x^2$, is also a function of x and its derivative may be found; so that from

$$y' = f'x = a \cdot 3x^2,$$

we have

$$y'' = f''x = a \cdot 3 \cdot 2x,$$

which is the derivative of y', or the derivative of the derivative of y, or simply the second derivative of y. In like manner the third derivative of y would be indicated by y''' or $f'''x$, the fourth by y^{iv} or f^{iv}, &c.

PROPOSITION I.

To find the derivative of any function capable of being developed in integral additive powers of the variable.

Let the function be denoted by

$$y = fx = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots + a_nx^n + \ldots. \tag{242}$$

Increasing x by h and the function by k, we have

$$y + k = f(x+h) = a_0 + a_1(x+h)^1 + a_2(x+h)^2 + \ldots + a_n(x+h)^n + \ldots,$$

$$\therefore \quad k = f(x+h) - fx = a_1[(x+h)^1 - x^1] + a_2[(x+h)^2 - x^2] + \ldots$$
$$+ a_n[(x+h)^n - x^n] + \ldots,$$

and

$$\frac{k}{h} = \frac{f(x+h)-fx}{h} = a_1 \cdot \frac{(x+h)^1 - x^1}{h} + a_2 \cdot \frac{(x+h)^2 - x^2}{h} + \ldots$$
$$+ a_n \cdot \frac{(x+h)^n - x^n}{h} + \ldots;$$

but

$$\frac{(x+h)^1 - x^1}{h} = \frac{h}{h} = 1,$$

$$\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x + h,$$

$$\frac{(x+h)^3 - x^3}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + (3x+h)h,$$

so $\frac{(x+h)^4 - x^4}{h}$ will be found $= 4x^3 + Ph$, where P depends upon x and h; and we are led to infer that

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + Xh,$$

where X is such a function of x and h that Xh shall disappear when h is diminished to zero. But to prove this let us put

$$x + h = a,\ x = b, \text{ and } \therefore\ h = a - b\,;$$

we have $$\frac{(x+h)^1 - x^1}{h} = \frac{a-b}{a-b},\ \frac{(x-h)^2 - x^2}{h} = \frac{a^2-b^2}{a-b},$$

$$\frac{(x+h)^3 - x^3}{h} = \frac{a^3-b^3}{a-b},\ \&c.,\ \frac{(x+h)^n - x^n}{h} = \frac{a^n-b^n}{a-b}\,;$$

so that we fall upon the examples under (16), and we have only to solve the problem whether $a^n - b^n$ is divisible by $a - b$.

In order to this we execute the multiplication,

$$(a-b)\,(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + a^{n-4}b^3 + a^{n-5}b^4 + \ldots + ab^{n-2} + b^{n-1}),$$

and we find the product to be

$$\begin{aligned} &a^n + a^{n-1}b + a^{n-2}b^2 + a^{n-3}b^3 + a^{n-4}b^4 + \ldots + a^2b^{n-2} + ab^{n-1} \\ &\quad - a^{n-1}b - a^{n-2}b^2 - a^{n-3}b^3 - a^{n-4}b^4 - \ldots - a^2b^{n-2} - ab^{n-1} - b^n, \end{aligned}$$

which is equal to $a^n - b^n$;

$$\therefore\ \frac{a^n - b^n}{a-b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^2b^{n-3} + ab^{n-2} + b^{n-1}. \quad (243)$$

Hence, the difference of the same powers of any two quantities is divisible by the difference of the quantities themselves, and the quotient is homogeneous and one degree lower than the power, the leading quantity descending one degree each term and the following ascending by the same law.

If in (243) we now replace a and b by their values $x + h$ and x, we find

$$\frac{(x+h)^n - x^n}{h} = (x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + x^{n-1}\,;$$

whence the fourth equation becomes

$$\frac{k}{h} = \frac{f(x+h) - fx}{h} = a_1 + a_2\,(2x + h) + a_3\,(3x^2 + 3xh + h^2) + \ldots$$

$$+ a_n\,[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + \ldots + x^{n-1}]\,;$$

therefore, making $h = 0$ and observing that there are n terms in the expression $(x+h)^{n-1} + \ldots$, which all reduce to x^{n-1}, we get

$$y' = f'x = \left[\frac{k}{h}\right] = \left[\frac{f(x+h) - fx}{h}\right] = a_1 + a_2 \cdot 2x + a_3 \cdot 3x^2 \quad (244)$$

$+ a_4 \cdot 4x^3 + \ldots + a_n \cdot nx^{n-1}$; which is the derivative sought.

We perceive that (244) is derived from (242) by multiplying by the exponents of x in the several terms and decreasing them by

unity; and hence that the term a_0 independent of x, or, which is the same thing, the term $a_0 \cdot x^0$, disappears in (244); so that, if we were to pass from (244) back to (242), it would be necessary to introduce a term independent of x, or to add a constant quantity, which may be represented by C.

PROPOSITION II.

To find the Derivative of any real power of a variable.

Let $y = fx = Ax^a$ be the function, a being any whatever, whole or fractional, plus or minus, but not imaginary, not of the form $a = \sqrt[2n]{-c}$. Suppose, in the first place, that a is fractional and additive,

or $$a = +\frac{m}{n};$$

then $$y = fx = Ax^{\frac{m}{n}} = A\left(x^{\frac{1}{n}}\right)^m.$$

Now, if we substitute z for $x^{\frac{1}{n}}$, we get

$$y = f_1 z = Az^m, \text{ and } z = f_2 x = x^{\frac{1}{n}}.$$

Suppose then that, while x is increased by h, z receives the increment i, and, as a consequence, that y is augmented by k; we have

$$y + k = A(z+i)^m, \text{ and } z + i = (x+h)^{\frac{1}{n}}, \text{ or } (z+i)^n = x + h;$$

$$\therefore \quad k = A(z+i)^m - Az^m, \text{ and } (z+i)^n - z^n = h;$$

$$\therefore \frac{k}{i} = A \cdot \frac{(z+i)^m - z^m}{i}, \text{ and } \frac{i}{h} = \frac{i}{(z+i)^n - z^n} = \frac{1}{\frac{(z+i)^n - z^n}{i}};$$

$$\therefore \quad \frac{k}{h} = \frac{k}{i} \cdot \frac{i}{h} = A \cdot \frac{(z+i)^m - z^m}{i} = \frac{1}{\frac{(z+i)^n - z^n}{i}};$$

$$\therefore \quad \left[\frac{k}{h}\right] = \left[\frac{k}{i}\right] \cdot \left[\frac{i}{h}\right] = A \cdot mz^{m-1} \cdot \frac{1}{nz^{n-1}} = A \cdot \frac{m}{n} \cdot z^{m-n}$$

$$= A \cdot \frac{m}{n}\left(x^{\frac{1}{n}}\right)^{m-n} = A \cdot \frac{m}{n} \cdot x^{\frac{m}{n}-1},$$

or* $$y'_{y=fx} = y'_{y=f_1 z} \cdot z'_{z=f_2 x} = A \cdot \frac{m}{n} \cdot x^{\frac{m}{n}-1}.$$

---

* $y'_{y=fx}$, read, the derivative of y, y being a function of x.

Here it is worthy of remark that we have found the derivative of y a function of x, by taking the product of the derivative of y a function of z, and that of z a function of x. We observe the same rule, then, for the derivative, whether the exponent be fractional or integral, we multiply by the exponent, and diminish it by unity.

Let a be now taken subtractive, and either integral or fractional, or let

$$a = -r,$$

where r is either a whole number or a fraction; we have

$$y = Ax^a = Ax^{-r};$$

$$\therefore \quad y + k = A(x+h)^{-r},$$

and $\quad k = A(x+h)^{-r} - Ax^{-r};$

$$\therefore \quad \frac{k}{h} = A \cdot \frac{(x+h)^{-r} - x^{-r}}{h} = A \cdot \frac{(x+h)^{-r} - x^{-r}}{h} \cdot \frac{(x+h)^r x^r}{(x+h)^r x^r}$$

$$= A \cdot \frac{(x+h)^{r-r}x^r - x^{r-r}(x+h)^r}{(x+h)^r x^r h} = A \cdot \frac{x^r - (x+h)^r}{(x+h)^r x^r h}$$

$$= -A \cdot \frac{\dfrac{(x+h)^r - x^r}{h}}{(x+h)^r x^r};$$

$$\therefore \quad y' = f'x = \left[\frac{k}{h}\right] = -A \cdot \frac{rx^{r-1}}{x^{2r}} = A \cdot -rx^{-r-1},$$

the derivative sought. Hence,

RULE.

To find the Derivative of a variable, affected by any exponent whatever that is not imaginary; multiply the variable by its exponent, and decrease that exponent by unity. (245)

Thus, the derivative of x^2 is $2x^{2-1}$, of x^3 is $3x^2$, of x^4 is $4x^3$; of $x^{\frac{2}{3}}$ is $\frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3} \cdot \frac{1}{x^{\frac{1}{3}}}$, of $x^{\frac{7}{5}}$ is $\frac{7}{5} \cdot x^{\frac{7}{5}-1} = \frac{7}{5} \cdot x^{\frac{2}{5}}$, of x^{-10} is $-10x^{-10-1} = -10 \cdot \frac{1}{x^{11}}$; of Ax^a is Aax^{a-1}; of x is $1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$; of A, $(= Ax^0)$, it is $A \cdot 0 \cdot x^{0-1} = A \cdot 0 \cdot \frac{1}{x} = 0.$

PROPOSITION III.

The Derivative of any function, capable of being developed in real powers of the variable, will be found by multiplying in each term by the exponent, and decreasing it one. (246)

CONVERSELY: *if a derived function be developed in powers of the variable, we shall return to the original function, by increasing the exponents by unity, and dividing by the exponents thus augmented, taking care to add a constant.*

Thus, let y be such a function of x, or depend upon x in such way, that we have

$$y = fx = Ax^a + Bx^b + Cx^c + \ldots, \tag{247}$$

where A, a; B, b; C, c; &c., may be any real quantities, + or −, whole or fractional, we have (245)

$$y' = f'x = \left[\frac{k}{h}\right] = Aax^{a-1} + Bbx^{b-1} + Ccx^{c-1} + \ldots. \tag{248}$$

We observe that if $a = 0$, or there be a constant term, $Ax^0 = A$, in (247), it will disappear in (248), since Aax^{a-1} then becomes $A \cdot 0 \cdot \frac{1}{x} = 0$; therefore, in passing back from (248) to (247) we must add a constant for that which may have disappeared. We shall see how this constant will be determined in any particular case by the nature of the problem.

To illustrate, suppose we have found the derivative

$$y' = 3x - \tfrac{4}{5}x^{\frac{1}{5}} + 7,$$

returning to the function (246) we obtain

$$y = \tfrac{3}{2}x^2 - \tfrac{5}{6} \cdot \tfrac{4}{5}x^{\frac{6}{5}} + 7x + \textit{constant}.$$

Whenever, then, we can find the derived function developed in powers of the variable, there will be no difficulty in ascertaining the primitive function.

PROPOSITION IV.

The Derivatives of equal functions, depending upon the same variable, are themselves equal. (249)

This is manifest from the nature of the operation; thus, if we have any two functions of x, such that

$$Fx = fx,$$

whatever may be the value of x, then are the functions equal when x is increased by h, or when x becomes $x+h$, and we have

$$F(x+h)=f(x+h),$$

from which subtracting the first equation, there results

$$F(x+h)-Fx=f(x+h)-fx,$$

which, divided by h, gives

$$\frac{F(x+h)-Fx}{h}=\frac{f(x+h)-fx}{h};$$

and this equality, having been obtained independently of any supposition in regard to the magnitude of h, is true for all values of h, and therefore true when h becomes less than any assignable quantity, or when h is diminished to an equality with zero.

Therefore, $$\left[\frac{F(x+h)-Fx}{h}\right]=\left[\frac{f(x+h)-fx}{h}\right],$$

or $$F'x=f'x.$$

Thus, if

$$A_0+A_1\cdot x^1+A_2\cdot x^2+\ldots=a_0+a_1\cdot x^1+a_2\cdot x^2+\ldots,$$

then shall we find

$$A_1+A_2\cdot 2x+\ldots=a_1+a_2\cdot 2x+\ldots.$$

We observe that if the last equation be multiplied by x, and subtracted from the preceding, we obtain

$$A_0-A_2\cdot x^2-A_3\cdot 2x^3-\ldots=a_0-a_2\cdot x^2-a_3\cdot 2x^3-\ldots,$$

from which the first power of x is eliminated; and, in like manner, we might eliminate in succession the terms affected by x^2, x^3, &c., to the last, if the series were finite, and to a term less than any assignable quantity, if infinite and convergent, that is, if x were less than unity; hence we should obtain $A_0=a_0$, and, pursuing the same process, $A_1=a_1$, $A_2=a_2$, and so on, which is in accordance with (65).

Scholium I. The student will not embarrass himself with unnecessary difficulty in reference to the symbol

$$\left[\frac{k}{h}\right], \text{ or its equivalent } \left[\frac{f(x+h)-fx}{h}\right],$$

by supposing it to signify nothing more than is indicated by $\frac{0}{0}$ in itself considered. It is true that the numerator as well as the denominator in $\left[\frac{k}{h}\right]$, is $=0$; but we are not to infer from this, either that the quotient, $\left[\frac{k}{h}\right]$, is equal to unity, or that one 0 can

have a different value from another 0, in order to make the quotient, $\frac{0}{0}$, represent different quantities; but that $\left[\frac{k}{h}\right]$ is simply a SYMBOL of a determinate operation, whereby we have derived one quantity from another. We are not, therefore to separate the k from the h as if they were *determinate* quantities, and as such could be managed separately, nor even to remove the brackets; the whole expression, $\left[\frac{k}{h}\right]$, let it be remembered, is one indivisible symbol, the sign of a process, it does not so much signify what the result is as how it is obtained. In any function, depending upon x, we give to x an increment h, we subtract the unaugmented function away, we divide by the increment h, and, in the quotient, eliminate h by reducing h to 0. Nor is this a singular case; all algebra is but a system of symbols, a mathematical language, telling what is to be done, and showing how results are obtained. Thus, ab, $a : b$, a^2, $\sqrt{a}$, are not simply the representatives of quantities, but indicate the operations, multiplication, division, involution, evolution, whereby certain quantities are obtained.

Scholium II. It is obvious that the derived functions will differ from each other according as the primitive functions, from which they are derived, are different.

SECTION SECOND.

The Binomial, Logarithmic, Interpolating, and Exponential Theorems.

PROPOSITION I.

It is required to develope $(a+x)^n$ *in powers of* x, n *being any real quantity, plus or minus, integral or fractional.*

First, suppose n to be plus and integral, that is, any of the numbers, 1, 2, 3, 4, 5, ... It is manifest that the development of $(a+x)^n$ can contain no other than plus integral powers of x, and must contain these, since the multiplication of whole positive

powers by whole positive powers gives whole positive powers; thus,

$$(a+x)^2 = (a+x)(a+x) = a^2 + 2ax + x^2,$$
$$(a+x)^3 = (a+x)(a^2+2ax+x^2), \text{ \&c.};$$

hence, $(a+x)^n$ must $= A_0 + A_1 \cdot x^1 + A_2 \cdot x^2 + A_3 \cdot x^3 + \ldots$
$$+ A_m \cdot x^m + \ldots, \quad (a)$$

arranging x in ascending powers, and denoting any term by $A_m \cdot x^m$.

In order to determine the coëfficients $A_0, A_1, A_2, \ldots, A_m$, of the several powers of x, it will be necessary to take the derivative of (a); therefore, to find the derivative of the first member $(a+x)^n$, put

$$y = (a+x)^n = z^n, \text{ and } \therefore\ z = a + x;$$
$$\therefore\quad y + k = (z+i)^n, \text{ and } z + i = a + x + h;$$
$$\therefore\quad \frac{k}{i} = \frac{(z+i)^n - z^n}{i}, \text{ and } \frac{i}{h} = 1;$$
$$\therefore\quad \frac{k}{h} = \frac{k}{i} \cdot \frac{i}{h} = \frac{(z+i)^n - z^n}{i} \cdot 1;$$
$$\therefore\quad y'_{y=fx} = \left[\frac{k}{h}\right] = \left[\frac{k}{i}\right] \cdot \left[\frac{i}{h}\right] = nz^{n-1} = n(a+x)^{n-1},$$

whence the same rule holds for the derivative of $(a+x)^n$ as for x^n, substituting $a + x$ for x.

The derivative of equation (a) gives us then (249) the equality

$$n(a+x)^{n-1} = A_1 + A_2 \cdot 2x + A_3 \cdot 3x^2 + A_4 \cdot 4x^3 + \ldots$$
$$+ A_m \cdot mx^{m-1} + \ldots; \quad (b)$$

Taking the derivative of (b), we have

$$n(n-1)(a+x)^{n-2} = A_2 \cdot 2 \cdot 1 + A_3 \cdot 3 \cdot 2x + A_4 \cdot 4 \cdot 3x^2$$
$$+ \ldots + A_m \cdot m(m-1)x^{m-2} + \ldots; \quad (c)$$

the third derivative gives

$$n(n-1)(n-2)(a+x)^{n-3} = A_3 \cdot 3 \cdot 2 \cdot 1 + A_4 \cdot 4 \cdot 3 \cdot 2x$$
$$+ \ldots + A_m \cdot m(m-1)(m-2)x^{m-3} + \ldots; \quad (d)$$

the fourth derivative will be found

$$n(n-1)(n-2)(n-3)(a+x)^{n-4} = A_4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + \ldots$$
$$+ A_m \cdot m(m-1)(m-2)(m-3)x^{m-4} + \ldots; \quad (e)$$

therefore, observing the law of derivation in (a), (b), (c), (d), (e) $\ldots$, and proceeding to the mth derivative, we obtain

$$n(n-1)(n-2)(n-3)(n-4)\ldots(n-m+1)(a+x)^{n-m}$$
$$= A_m \cdot m(m-1)(m-2)(m-3)\ldots \cdot 3 \cdot 2 \cdot 1 + \ldots. \quad (l)$$

From the last derivative we observe that, if n be a whole additive number, m can never exceed n; for if m were $= n+1$, $n-m+1$ would be $=0$, and the $(n+1)$th derivative, and consequently all after it, would disappear. This is as it should be, since it is obvious that the nth power of $(a+x)$ cannot give any term in the development of a higher degree than n.

As the coëfficients $A_0, A_1, A_2, \ldots, A_m, \ldots$, are independent of x, we may make $x=0$ in (a), (b), (c), (d), (e), ..., (l), ..., and we obtain

$$a^n = A_0,\ na^{n-1} = A_1,\ n(n-1)a^{n-2} = A_2 \cdot 2 \cdot 1,$$
$$n(n-1)(n-2)a^{n-3} = A_3 \cdot 3 \cdot 2 \cdot 1,$$
$$n(n-1)(n-2)(n-3)^{n-4} = A_4 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \ldots$$
$$n(n-1)(n-2)(n-3)(n-4)\ldots(n-m+1)a^{n-m}$$
$$= A_m \cdot m(m-1)(m-2)\ldots \cdot 3 \cdot 2 \cdot 1.$$

Substituting the values of $A_0, A_1, A_2 \ldots, A_n$, drawn from these equations in (a), we obtain

NEWTON'S BINOMIAL THEOREM.

$$(a+x)^n = a^n + na^{n-1} \cdot x + n(n-1)a^{n-2} \cdot \frac{x^2}{2} \qquad (250)$$
$$+ n(n-1)(n-2)a^{n-3} \cdot \frac{x^3}{2 \cdot 3} + \ldots$$
$$+ n(n-1)(n-2)(n-3)\ldots \cdot (n-m+1)a^{n-m} \cdot \frac{x^m}{1 \cdot 2 \cdot 3 \ldots m} + \ldots.$$

Since the same rules of addition, subtraction, multiplication, division, and derivation are applicable, whether the exponents be plus or minus, integral or fractional, we might infer that (250) would hold in all cases, whatever might be the character of n. But to prove it observe that if n be any real quantity [not imaginary], there can be no other than real powers of x in the development, such as $x^0, x^1, x^2, x^3, \ldots x^m, \ldots x^{\frac{e}{f}}, \ldots x^{-r}, x^{-s}, x^{-t} \ldots$.

Therefore the form

$$(a+x)^n = A_0 \cdot x^0 + A_1 \cdot x^1 + \ldots + A_{\frac{e}{f}} \cdot x^{\frac{e}{f}} + \ldots + A_{-r} \cdot x^{-r}$$
$$+ A_{-s} \cdot x^{-s} + A_{-t} \cdot x^{-t} + \ldots,$$

must at least be sufficiently general, Suppose now that the minus powers are arranged in a descending order, that is, that $r > s$, $s > t \ldots$; $\therefore$ multiplying by x^r, we have

$$(a+x)^n \cdot x^r = (A_0 + A_1 \cdot x^1 + \dots + A_{\frac{e}{f}} \cdot x^{\frac{e}{f}} + \dots)\, x^r$$
$$+ A_{-r} + A_{-s} \cdot x^{r-s} + A_{-t} \cdot x^{r-t} + \dots,$$

where it is obvious that all the powers of x are plus; $\therefore$ making $x=0$, there results

$$0 = 0 + A_{-r} + 0 + \dots, \quad \therefore A_{-r} = 0;$$

and in the same way it may be shown that $A_{-s}=0$, $A_{-t}=0, \dots$; that is, the expansion contains no minus exponents. Rejecting the minus powers and taking the mth derivative, we have

$$n(n-1)(n-2)\dots\cdot(n-m+1)(a+x)^{n-m} = A_m \cdot m(m-1)$$
$$(m-2)\dots\cdot 3\cdot 2\cdot 1 + A_{m+1}\cdot(m+1)\,m(m-1)\dots 2x + \dots$$
$$+ A_{\frac{e}{f}} \cdot \frac{e}{f}\left(\frac{e}{f}-1\right)\left(\frac{e}{f}-2\right)\dots\cdot\left(\frac{e}{f}-m+1\right)x^{\frac{e}{f}-m} + \dots.$$

The order of this derivative may be any whatever, if n be minus or fractional, since in this case the coëfficient $n-m+1$ can never become $=0$, m being a whole additive number; we can therefore take m so great that the exponent $\frac{e}{f}-m$ shall be minus, $\left(\frac{e}{f}-m\right.$ can never $=0$, m being $=1, 2, 3, \dots$). But

$$n(n-1)\dots\cdot(n-m+1)(a+x)^{n-m}$$

cannot have a minus exponent in its development, for the same reason that $(a+x)^n$ has none. It follows that the development of $(a+x)^n$ will not be affected by any fractional power of x, whatever n may be; (250) is therefore true for all the real values of n, since for all such values the coëfficients of the expansion have been determined. It appears, however, that if n be either minus or fractional, the series will never terminate.

As the form of (250) is independent of the value of a or x, we may change x into $-x$, doing which and observing that the terms affected by the odd powers of x will be minus, we obtain

$$(a-x)^n = a^n - na^{n-1}\cdot x + n(n-1)\cdot\frac{x^2}{2} - n(n-1)(n-2)$$
$$\cdot\frac{x^3}{2\cdot 3} +, -, \dots. \tag{251}$$

If in (250) and (251) we make $a=1$ and $x=1$, and remember that all the powers of 1 are 1, we get

$$2^n = 1 + n + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{2\cdot 3} + \dots,$$

and $$0 = 1 - n + \frac{n(n-1)}{2} - \frac{n(n-1)(n-2)}{2\cdot 3} +, -, \dots.$$

Thus it appears (1°) that the sum of the coëfficients of any binomial power is equal to 2 raised to the same power, and (2°) that the sum of the coëfficients of any binomial power taken alternately plus and minus is equal to zero. This furnishes a convenient method of verifying an involution. For example,

$(a+x)^1 = 1 \cdot a + 1 \cdot x$, $1+1=2=2^1$, and $+1-1=0$;

$(a+x)^2 = a^2+2ax+x^2$, $1+2+1=4=2^2$, and $+1-2+1=0$;

$(a+x)^3 = a^3+3a^2x+3ax^2+x^3$, $1+3+3+1=8=2^3$, and $1-3+3-1=0$.

Expand $(a+x)^4$, $(a+x)^5$, $(a+x)^6$, $(a+x)^7$, by aid of (250), and $(a-x)^4$, $(a-x)^5$, $(a-x)^6$, $(a-x)^7$, by aid of (251), and verify.

If in (250) and (251), we make $n=\frac{1}{r}$, there results

$$(a+x)^{\frac{1}{r}} = a^{\frac{1}{r}} + \frac{1}{r}a^{\frac{1}{r}-1} \cdot x + \frac{1}{r}\left(\frac{1}{r}-1\right)a^{\frac{1}{r}-2} \cdot \frac{x^2}{2}$$
$$+\frac{1}{r}\left(\frac{1}{r}-1\right)\left(\frac{1}{r}-2\right)a^{\frac{1}{r}-3} \cdot \frac{x^3}{2 \cdot 3} + \dots,$$

or
$$(a+x)^{\frac{1}{r}} = a^{\frac{1}{r}} + \frac{x}{ra^{\frac{r-1}{r}}} - \frac{(r-1)x^2}{r^2 \cdot 2a^{\frac{2r-1}{r}}} + \frac{(r-1)(2r-1)x^3}{r^3 \cdot 2 \cdot 3a^{\frac{3r-1}{r}}}$$
$$-\frac{(r-1)(2r-1)(3r-1)x^4}{r^4 \cdot 2 \cdot 3 \cdot 4a^{\frac{4r-1}{r}}} +, -, \dots, \tag{252}$$

and
$$(a-x)^{\frac{1}{r}} = a^{\frac{1}{r}} - \frac{x}{ra^{\frac{r-1}{r}}} - \frac{(r-1)x^2}{r^2 \cdot 2a^{\frac{2r-1}{r}}} - \frac{(r-1)(2r-1)x^3}{r^4 \cdot 2 \cdot 3 \cdot 4a^{\frac{4r-1}{r}}} - \dots. \tag{253}$$

Making $r=2, 3$, in (252), (253), we get

$$(a+x)^{\frac{1}{2}} = a^{\frac{1}{2}} + \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{2^2 \cdot 2a^{\frac{3}{2}}} + \frac{3x^3}{2^3 \cdot 2 \cdot 3a^{\frac{5}{2}}}$$
$$-\frac{3 \cdot 5x^4}{2^4 \cdot 2 \cdot 3 \cdot 4a^{\frac{7}{2}}} +, -, \dots, \tag{254}$$

$$(a-x)^{\frac{1}{2}} = a^{\frac{1}{2}} - \frac{x}{2a^{\frac{1}{2}}} - \frac{x^2}{2^2 \cdot 2a^{\frac{3}{2}}} - \frac{3x^3}{2^3 \cdot 2 \cdot 3a^{\frac{5}{2}}} -, \dots; \tag{255}$$

$$(a+x)^{\frac{1}{3}} = a^{\frac{1}{3}} + \frac{x}{3a^{\frac{2}{3}}} - \frac{2x^2}{3^2 \cdot 2a^{\frac{5}{3}}} + \frac{2 \cdot 5x^3}{3^3 \cdot 2 \cdot 3a^{\frac{8}{3}}}$$
$$-\frac{2 \cdot 5 \cdot 8x^4}{3^4 \cdot 2 \cdot 3 \cdot 4a^{\frac{11}{3}}} +, -, \dots, \tag{256}$$

$$(a-x)^{\frac{1}{3}}=a^{\frac{1}{3}}-\frac{x}{3a^{\frac{2}{3}}}-\frac{2x^2}{3^2\cdot 2a^{\frac{5}{3}}}-,\ldots. \qquad (257)$$

In (254) making $a=c^2$, we find

$$(c^2+x)^{\frac{1}{2}}=c+\frac{x}{2c}-\frac{x^2}{2^2\cdot 2c^3}+\frac{3x^3}{2^3\cdot 2\cdot 3c^5}$$
$$-\frac{3\cdot 5x^4}{2^4\cdot 2\cdot 3\cdot 4c^7}+,-,\ldots; \qquad (258)$$

from which, substituting z^2 for x, we obtain

$$(c^2+z^2)^{\frac{1}{2}}=c+\frac{z^2}{2c}-\frac{z^4}{2^2\cdot 2c^3}+\frac{3z^6}{2^3\cdot 2\cdot 3c^5}$$
$$-\frac{3\cdot 5z^8}{2^4\cdot 2\cdot 3\cdot 4c^7}+,-,\ldots; \qquad (259)$$

changing x into $-x$ in (258),

$$(c^2-x)^{\frac{1}{2}}=c-\frac{x}{2c}-\frac{x^2}{2^2\cdot 2c^3}-\frac{3x^3}{2^3\cdot 2\cdot 3c^5}$$
$$-\frac{3\cdot 5x^4}{2^4\cdot 2\cdot 3\cdot 4c^7}-,\ldots; \qquad (260)$$

changing x into z^2

$$(c^2-z^2)^{\frac{1}{2}}=c-\frac{z^2}{2c}-\frac{z^4}{2^2\cdot 2c^3}-\frac{3z^6}{2^3\cdot 2\cdot 3c^5}-,\ldots. \qquad (261)$$

The student may operate like changes upon (256), and investigate analogous forms for higher roots, as the 4th, 5th, &c. These expansions may be employed for the extraction of the roots of numbers. Thus, let the square root of 101 be required. We have (258)

$$101^{\frac{1}{2}}=(10^2+1)^{\frac{1}{2}}=10+\tfrac{1}{20}-\tfrac{1}{8000}+\tfrac{1}{1600000}-\tfrac{5}{1280000000}$$
$$=10{\cdot}0509381211.$$

Find $\sqrt{102}$, $\sqrt{103}$, $\sqrt{104}$; $\sqrt{99}$, $\sqrt{98}$, $\sqrt{97}$, $\sqrt[3]{1001}$, &c.

We may put (250) under a new form, for dividing by a^n, there results,

$$\left(1+\frac{x}{a}\right)^n=1+n\cdot\frac{x}{a}+n(n-1)\cdot\frac{x^2}{2a^2}$$
$$+n(n-1)(n-2)\cdot\frac{x^3}{2\cdot 3a^3}+\ldots, \qquad (262)$$

or

$$(1+v)^n=1+nv+n(n-1)\cdot\frac{v^2}{2}$$
$$+n(n-1)(n-2)\cdot\frac{v^3}{2\cdot 3}+\ldots,\text{ putting } v=\frac{x}{a}. \qquad (262)$$

LOGARITHMS.

Definition. Let $a^y = x$, (263)
then is y denominated the *Logarithm* of x. The constant a is called the base of the system.

PROPOSITION II.

The Logarithm of a product, consisting of several factors, is equal to the sum of the Logarithms of those factors. (264)

For, let $x = x_1, x_2, x_3, \ldots$; $y = y_1, y_2, y_3, \ldots$;

then (263)

$$a^{y_1} = x_1,$$
$$a^{y_2} = x_2,$$
$$a^{y_3} = x_3,$$
&c., &c.

$$\therefore \quad a^{y_1} \cdot a^{y_2} \cdot a^{y_3} \cdot \ldots = a^{y_1+y_2+y_3+\ldots} = x_1 \cdot x_2 \cdot x_3 \cdot \ldots,$$

or* $L(x_1 \cdot x_2 \cdot x_3 \ldots) = (y_1 + y_2 + y_3 + \ldots) = Lx_1 + Lx_2 + Lx_3 + \ldots$. Q. E. D.

Cor. 1. The logarithm of the nth power of any number is equal to n times the logarithm of the number itself. (265)

For, making $x_1 = x_2 = x_3 = \ldots$, we have

$$L(x_1 \cdot x_1 \cdot x_1 \cdot \ldots [n]) = Lx_1 + Lx_1 + Lx_1 + \ldots,$$

or, $L(x_1^n) = nLx_1$. (265)

Cor. 2. The logarithm of the nth root of any number, is equal to the nth part of the logarithm of the number itself. (266)

Cor. 3. The logarithm of a fraction is equal to the logarithm of the numerator diminished by the logarithm of the denominator. (267)

For, if we divide $a^{y_1} = x_1$,

by $a^{y_2} = x_2$,

we have $a^{y_1 - y_2} = \dfrac{x_1}{x_2}$;

or $L\left(\dfrac{x_1}{x_2}\right) = y_1 - y_2 = Lx_1 - Lx_2$.

Scholium. We perceive that addition of logarithms corresponds to multiplication of numbers, subtraction to division, multiplication to involution, and division to evolution. We have, then, only to possess a "*Table of Logarithms,*" calculated to a given base,

---

* L, logarithm of.

say $a = 10$, in order to perform numerical operation with remarkable facility. Thus, to obtain the cube root of 2, nothing more would be necessary than to take the logarithm of 2, divide by 3, and seek the corresponding number from the table.

Again, if $\quad 3 = 2^z,$

then (265) $\quad L3 = L(2^z) = zL2;$

$\therefore \quad z = \dfrac{L3}{L2};$

whereby we are enabled to solve, with the utmost facility, a numerical equation in which the exponent is the unknown quantity.

PROPOSITION III.

A number being given, it is required to find a form by which we may calculate its logarithm.

Since the relation $a^y = x$, gives y a function of x, let us endeavor to expand y in terms of x. To this end we proceed to determine the derivative of y.

Changing x into $x + h$ and y into $y + k$, we have

$$a^{y+k} = x + h;$$

subtracting $\quad a^y = x,$

there results $\quad a^y \cdot a^k - a^y = h,$

or $\quad h = a^y(a^k - 1);$

in which, substituting x for a^y and $1 + b$ for a, in order to subject a^k to the influence of the Binomial Theorem, we have (262)

$$h = x\,[(1 + b)^k - 1] = x\,[1 + kb + k(k-1) \cdot \frac{b^2}{2} + k(k-1)(k-2) \cdot \frac{b^3}{2 \cdot 3} + \ldots - 1];$$

dividing k by each member of this equation, observing that $1, -1$, cancel each other, and that k then becomes a factor common to the numerator and denominator of the second fraction, there results

$$\frac{k}{h} = \frac{1}{x\left[b + (k-1) \cdot \frac{b^2}{2} + (k-1)(k-2) \cdot \frac{b^3}{2 \cdot 3} + \ldots\right]};$$

from which, observing $k = 0$ when h becomes $= 0$, we have

$$y' = \left[\frac{k}{h}\right] = \frac{1}{\left[b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} +, -, \ldots\right]x} = M \cdot \frac{1}{x} = Mx^{-1}, \quad (268)$$

putting $$M = \frac{1}{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} +, -, \ldots} \qquad (269)$$

$$= \frac{1}{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} +, -, \ldots}.$$

Applying the rule of (246), in order to return from the derivative (268) to the primitive function, we fall upon the equation

$$y = M \cdot \frac{x^{-1+1}}{-1+1} + constant = \frac{M}{0} + constant,$$

from which x disappears, and nothing can be inferred—save that *the logarithm of a number cannot be developed in terms of the number simply.*

But if we make y the logarithm of $1 + x$ instead of x, and repeat the above operation, from

$$a^y = 1 + x,$$

we obtain $$y' = M(1+x)^{-1},$$

or (262), $$y' = M\left[1 + (-1)x + (-1)(-1-1)\frac{x^2}{2} + (-1)(-1-1)\right.$$

$$\left.(-1-2) \cdot \frac{x^3}{2 \cdot 3} + \ldots\right] = M[1 - x + x^2 - x^3 + x^4 - x^5 +, -, \ldots];$$

from which, returning to the function, we get

$$y = M[x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 +, -, \ldots] + constant.$$

In order to determine the constant, we observe that x and y vanish together, since $y = 0$ in $a^y = 1 + x$, gives $1 + x = a^0 = 1$, and $\therefore x = 0$; therefore, substituting these corresponding values of y and x, we get

$$0 = M \cdot 0 + constant, \; \therefore constant = 0, \text{ and}$$

$$L(1+x) = y = M[x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6 +, -, \ldots], \qquad (270)$$

a *logarithmic* series, in which the logarithm of any number $1 + x$ is expressed in terms of a number less by unity, x.

The constant, M, depending upon the base, a, (269), is denominated the *Modulus* of the system. Taking a different base, a, we obtain a new modulus,

$$M_2 = \frac{1}{(a_2 - 1) - \frac{(a_2-1)^2}{2} +, -, \ldots};$$

and the logarithm of $1+x$, derived from $a_2^y = 1+x$, becomes (270)

$$L_2(1+x) = M_2[x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 +, -, \ldots],$$

by which, dividing (270), there results

$$\frac{L\,(1+x)}{L_2(1+x)} = \frac{M}{M_2}; \text{ hence,}$$

Cor. 1. The moduli in different systems are to each other (271) as the logarithms of any given number in those systems.

If we make the modulus equal to unity, we have the logarithms employed by *Lord Napier*, a Scottish nobleman, who was the inventor of this admirable system of numbers; and, if we assume 10 for the base, we have the logarithms in common use, or those of Briggs. It is customary to indicate a Napierian logarithm by a small l, and a common logarithm by the abbreviation *log*. Adopting this notation, and observing that $10^y = 1+x$ gives $10^1 = 10$, or $\log. 10 = 1$, we have (271)

$$\frac{\log. 10}{l\,10} = \frac{M_{a-10}}{1}, \text{ or } M_{a-10} = \frac{1}{l\,10}. \qquad (272)$$

We have, then, only to make the modulus one in (270), and thereby calculate the Napierian logarithm of 10, in order to determine the modulus, M_{a-10}, of the common system; but, by a little artifice, we may convert this series into one more rapidly convergent, and therefore better adapted to our purpose.

In (270) changing x into $-x$, we have

$$L(1-x) = M(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \ldots),$$

which, subtracted from (270), gives

$$L(1+x) - L(1-x) = 2M(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots), \qquad (273)$$

or (267), $$L\left(\frac{1+x}{1-x}\right) = 2M(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \ldots). \qquad (274)$$

And this series will be expressed more conveniently by putting, as Borda has done,

$$\frac{1+x}{1-x} = \frac{m}{n}, \text{ and } \therefore x = \frac{m-n}{m+n}; \text{ whence}$$

$$L\left(\frac{m}{n}\right) = 2M\left[\left(\frac{m-n}{m+n}\right)^1 + \tfrac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \tfrac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \ldots\right]. \qquad (275)$$

Making $M = 1$, $m = 2$, $n = 1$, in order to calculate the Napierian logarithm of 2, we find

$$l(\tfrac{2}{1}) = l2 = 2[\tfrac{1}{3} + \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 + \ldots],$$

where we have only to divide successively by $9[=3^2]$ and the odd numbers 3, 5, 7, ..., as follows:

3	2·00000000		
9	·66666667	1	·66666667
9	·07407407	3	·02469136
9	·00823045	5	·00164609
9	·00091449	7	·00013064
9	·00010161	9	·00001129
9	·00001129	11	·00000103
9	·00000125	13	·00000009
9	·00000014	15	·00000001

$$\therefore\ l2 = {\cdot}69314718$$

Having thus obtained the Napierian logarithm of 2, $l4 = l(2 \cdot 2) = l2 + l2 = 2l2 = 1{\cdot}38629436$; and putting $m = 5$, $n = 4$,

$$l\tfrac{5}{4} = 2[\tfrac{1}{9} + \tfrac{1}{3} \cdot \frac{1}{9^3} + \tfrac{1}{5} \cdot \frac{1}{9^5} + \ldots] = {\cdot}22314354;$$

Operation.

9	2·00000000		
81	·22222222	1	·22222222
81	274348	3	91449
81	3387	5	677
81	42	7	6
			·22314354

but $l\frac{5}{4} = l5 - l4$,

$\therefore\ l5 = l4 + {\cdot}22314354$
$= 1{\cdot}60943790$;

$\therefore\ l10 = l(2 \cdot 5) = l2 + l5$
$= 2{\cdot}30258508$;

$$\therefore\ (272),\qquad M_{a=10} = \frac{1}{l10} = 0{\cdot}43429448;$$

$$\therefore\ (275),\ \log.\left(\frac{m}{n}\right) = 0{\cdot}86858896\left[\left(\frac{m-n}{m+n}\right)^1 + \tfrac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \ldots\right], \tag{276}$$

$$\text{or } \log.\ m = \log.\ n + 0{\cdot}86858896\left[\left(\frac{m-n}{m+n}\right)^1 + \tfrac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \ldots\right].$$

Suppose, now, it were our object to construct a table extending from 1 to 10000, we should commence with 100; since, in calculating up to $10000 = 100 \cdot 100$ we should fall upon 200, 300, 400, &c., $= 2 \cdot 100$, $3 \cdot 100$, $4 \cdot 100$, &c., whereby the logarithms of 2, 3, 4, &c., would be determined.

Since $\quad \log.\ 100 = \log.\ (10^2) = 2\log.\ 10 = 2 \cdot 1 = 2$,

$$\therefore\ \log.\ 101 = \log.\ 100 + 0{\cdot}86858896\left(\frac{1}{201} + \ldots\right)$$

$$= 2 + \frac{0{\cdot}86858896}{201} = 2{\cdot}0043214;$$

$$\therefore \qquad \log.\, 102 = \log.\, 101 + \frac{\cdot 86858896}{203} = 2\cdot 0086002\,;$$

$$\therefore \qquad \log.\, 103 = \log.\, 102 + \frac{\cdot 86858896}{205} = 2\cdot 0128372\,;$$

&c. &c. &c.

After two consecutive logarithms have been obtained, the operation may be shortened; for, putting $m = n + 1$, we have

$$\log.\,(n+1) = \log.\, n + \frac{\cdot 86858896}{2n+1},$$

$$\log.\,(n+2) = \log.\,(n+1) + \frac{\cdot 86858896}{2n+3}\,;$$

$$\therefore \quad \log.\,(n+2) = 2\log.\,(n+1) - \log.\, n - \frac{1\cdot 73717793}{4(n+1)^2 - 1} \qquad (277)$$

Observe also that $4(2n+3)$ is the difference of two consecutive divisors, $4(n+1)^2 - 1$, $4(n+2)^2 - 1$.

Making $n = 100$, $\log.\, 102 = 2\log.\, 101 - \log.\, 100 - \dfrac{1\cdot 7371779}{4 \cdot 101^2 - 1}$.

Operation.

$$\left.\begin{array}{r} 10001 \\ +\ 200 \\ \hline 10201 \\ 4 \\ \hline 40803)1\cdot 7371779(426 \end{array}\right\} \therefore \left\{\begin{array}{r} +\,4\cdot 0086428 \\ -\,2\cdot 0000000 \\ -\qquad 426 \\ \hline \therefore\ \log.\, 102 = 2\cdot 0086002 \end{array}\right.$$

$n = 101$, $\therefore \log.\, 103 = 2\log.\, 102 - \log.\, 101 - \dfrac{1\cdot 7371779}{40803 + 4(202+3)}$.

Operation.

$$\left.\begin{array}{r} 40803 \\ 820 \\ \hline 41623)1\cdot 7371779(417 \end{array}\right\} \therefore \left\{\begin{array}{r} +\,4\cdot 0172004 \\ -\,2\cdot 0043214 \\ -\qquad 417 \\ \hline \therefore\ \log.\, 103 = 2\cdot 0128373 \end{array}\right.$$

The student should continue the computation.

Before terminating this problem, we will make one other transformation whereby the calculation of logarithms will be rendered far more rapid.

For small numbers, Borda puts*

$$m=(p-1)^2(p+2)=p^3-3p+2,$$
$$n=(p+1)^2(p-2)=p^3-3p-2;$$

$\therefore \quad m-n=4,$

and $\quad m+n=2p^3-6p;$

therefore, substituting in (275) and making $M=1$, we have

$$l\left(\frac{(p-1)^2(p+2)}{(p+1)^2(p-2)}\right)=2\left[\left(\frac{2}{p^3-3p}\right)^1+\tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\tfrac{1}{5}\left(\frac{2}{p^3-3p}\right)^5+\ldots\right],$$

or $\quad 2l(p-1)+l(p+2)-2l(p+1)-l(p-2)$

$$=2\left[\left(\frac{2}{p^3-3p}\right)^1+\tfrac{1}{3}\left(\frac{2}{p^3-3p}\right)^3+\ldots\right].$$

Making $p=5, 6, 7, 8$, successively, we obtain

$$2l2-3l3+l7=2[\tfrac{1}{55}+\tfrac{1}{3}(\tfrac{1}{55})^3+\ldots]=\cdot036367644171,$$
$$l2+2l5-2l7=2[\tfrac{1}{99}+\tfrac{1}{3}(\tfrac{1}{99})^3+\ldots]=\cdot020202707317,$$
$$-4l2-l5+4l3=2[\tfrac{1}{161}+\tfrac{1}{3}(\tfrac{1}{161})^3+\ldots]=\cdot012422519998,$$
$$-5l3+l5+2l7=2[\tfrac{1}{244}+\tfrac{1}{3}(\tfrac{1}{244})^3+\ldots]=\cdot008196767203;$$

from which eliminating $l7$, we have

$$5l2+2l5-6l3=\cdot092937995659,$$
$$l2+3l5-5l3=\cdot028399474520,$$
$$-4l2-l5+4l3=\cdot012422519998;$$

eliminating $l3$, $\quad -2l2+l5=\cdot223143551312,$

$$-16l2+7l5=\cdot175710498070;$$

whence $\quad l2=\cdot693147180557,$

$$l5=1\cdot609437912426;$$

$\therefore \quad l10=2\cdot302585093003.$

Had the above operation been extended sufficiently far, we should have found (272)

$$M_{a=10}=0\cdot43429\ 44819\ 03251\ 82765$$

which, introduced into the last transformation, gives, for the common logarithm,

$$\log.(p+2)=2\log.(p+1)+\log.(p-2)-2\log.(p-1)$$
$$+\cdot8685889638\left[\left(\frac{2}{p(p^2-3)}\right)+\tfrac{1}{3}\left(\frac{2}{p(p^2-3)}\right)^3+\ldots\right]. \quad (278)$$

Employing this form and imitating the above operation for the Napierian logarithms, $l2$, $l3$, $l5$, $l7$, the student will find†

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* Francœur, Mathématiques pures.

† The number of digits employed in the calculation should exceed, according to the nature of the operation, those which it is intended to retain in the results.

log. 1 = 0, [$10^0 = 1$.]
log. 2 = 0·30102 99957,
log. 3 = 0·47712 12547,
log. 4 = 0·60205 99913, [log. 4 = 2log. 2.]
log. 5 = 0·69897 00043,
log. 6 = 0·77815 12504, [log. 6 = log. 2 + log. 3.]
log. 7 = 0·84509 80400,
log. 8 = 0·90308 99870, [log. 8 = 3log. 2.]
log. 9 = 0·95424 25094, [log. 9 = 2log. 3.]
log. 10 = 1·00000 00000; [$10^1 = 10$.]
∴ log. 11 = 1·04139 26852, [$p = 9$.]

since

log. 11 = 2log. 10 + log. 7 − 2log. 8 + ·86... $[\frac{1}{881} + \frac{1}{3}(\frac{1}{881})^3 + ...]$;

log. 12 = 1·07918 12460, [log. 12 = log. 3 + log. 4.]
log. 13 = 1·11394 33523, [$p = 11$.]
log. 14 = log. 2 + log. 7, log. 15 = log. 3 + log. 5, log. 16 = 2log. 4,
log. 17 = 1·23044 89214, [p = what?] log. 18 = ?
log. 19 = 1·27875 36010, log. 20 = ? log. 21 = ? log. 22 = ?
log. 23 = 1·36172 78360, log. 24 = ? log. 25 = ? log. 26 = ?
log. 27 = ? log. 28 = ?
log. 29 = 1·46239 79979, log. 30 = ?
log. 31 = 1·4913616938, log. 32 = ? log. 33 = ? log. 34 = ?
log. 35 = ? log. 36 = ? log. 37 = ? log. 38 = ? log. 39 = ?
log. 40 = ? log. 50 = ? log. 60 = ? log. 100 = ? log. 1000 = ?
log. 10000 = ? log. $\frac{1}{10}$ = ? log. $\frac{1}{100}$ = ? log. ·001 = ?

As p increases, the series (278) increases in convergency very rapidly; so much so, that, when p is no greater than 102, the second term will have its first significant figure in the 18th decimal place; and, confining ourselves to 7 decimals, the last two only will have to be obtained by division.

If we confine ourselves to seven digits, after the computation has been made up to 1000, the remaining logarithms may be readily calculated from the table itself.

Taking the differences of five consecutive logarithms, as those of 100, 101, 102, 103, 104, and the differences of these differences or the second differences, and the third differences, we find

N.	Log.	1st Dif. (+)	2d Dif. (—)	3d Dif. (—)
100	2·0000000	43214	426	8
101	2·0043214	42788	418	9
102	2·0086002	42370	409	
103	2·0128372	41961		
104	2·0170333			

We observe here that the third differences, 8, 9, are nearly equal. We are naturally led to the following problem:

PROPOSITION IV.

It is required to determine what function y_x *is of* x, *when* y_x *depends upon* x *in such way that, if we attribute to* x *the particular values* $x = 0, 1, 2, 3, 4$, *the third differences of the corresponding values of the function,* $y_x = y_0, y_1, y_2, y_3, y_4$, *shall be equal to each other.*

If in the first power of x,
we make $x = 0, 1, 2, 3, 4$,
the first differences, 1, 1, 1, 1, are the same;
if $x = 0, 1, 2, 3, 4$,
x^2 gives $x^2 = 0, 1, 4, 9, 16$,
the first differences are, 1, 3, 5, 7,
and the second differences 2, 2, 2, are the same;

if $x = 0, 1, 2, 3, 4$,
x^3 gives $x^3 = 0, 1, 8, 27, 64$,
the first differences are, 1, 7, 19, 37,
the second differences are, 6, 12, 18,
and the third differences, 6, 6, are constant.
Hence we assume

$$y_x = A + Bx + Cx^2 + Dx^3.$$

Attributing to x the particular values 0, 1, 2, 3, indicating the corresponding particular values of y_x by y_0, y_1, y_2, y_3, taking the differences as above, and denoting the first difference, $y_1 - y_0$, of the first differences by D_1, the first difference of the second differences by D_2, the first of the third differences by D_3, (which, for the sake of distinction, may be called the *first*, *second*, and *third* differences,) we have,

$$\begin{array}{lll}
y_0 = A, & & \\
& D_1 = y_1 - y_0 = B + C + D, & \\
& & D_2 = C \cdot 2 + D \cdot 6, \\
y_1 = A + B + C + D, & & \\
& y_2 - y_1 = B + C \cdot 3 + D \cdot 7; & D_3 = D \cdot 6; \\
y_2 = A + B \cdot 2 + C \cdot 4 + D \cdot 8, & & C \cdot 2 + D \cdot 12; \\
& B + C \cdot 5 + D \cdot 19; & \\
y_3 = A + B \cdot 3 + C \cdot 9 + D \cdot 27; & &
\end{array}$$

$\therefore$ $D = \frac{1}{6}D_3$, $C = \frac{1}{2}(D_2 - D_3)$, $B = D_1 - \frac{1}{2}D_2 + \frac{1}{3}D_3$, and $A = y_0$;

hence $y_x = y_0 + (D_1 - \frac{1}{2}D_2 + \frac{1}{3}D_3)x + \frac{1}{2}(D_2 - D_3)x^2 + \frac{1}{6}D_3 \cdot x^3$, (279)

or $\quad y_x = y_0 + x[D_1 - \frac{1}{2}D_2 + \frac{1}{3}D_3 + x \cdot \frac{1}{2}(D_2 - D_3 + x \cdot \frac{1}{3}D_3)]$,

or $\quad y_x = y_0 + x \cdot \dfrac{D_1}{1} + x(x-1) \cdot \dfrac{D_2}{1 \cdot 2} + x(x-1)(x-2) \cdot \dfrac{D_3}{1 \cdot 2 \cdot 3}$,

is the function sought.

For any particular series of numbers, it will be advantageous to calculate beforehand the coëfficients of the several powers of x, and to attribute to the terms their proper signs. Thus, for logarithms, the second and third differences, D_2, D_3, being minus, we have

$$y_x = y_0 + x\left[D_1 + \tfrac{1}{2}D_2 - \tfrac{1}{3}D_3 - x\left(\frac{D_2 - D_3}{2} + x \cdot \frac{D_3}{6}\right)\right],$$

or $\quad y_x = y_0 + x[C_1 - x(C_2 + x C_3)]$, (280)

putting $\quad C_1 = D_1 + \frac{1}{2}D_2 - \frac{1}{3}D_3$, $C_2 = \frac{1}{2}(D_2 - D_3)$, $C_3 = \frac{1}{6}D_3$.

Let us make an application by requiring the logarithm of 100·345. We have

$x = {}$·345,	$D_1 = {}$·0043214,	$\therefore$ $C_1 = {}$·0043424,
$y_0 = 2$·0000000;	$D_2 = {}$·0000426,	$C_2 = {}$·0000209,
	$D_3 = {}$·0000008;	$C_3 = {}$·0000001$\frac{1}{3}$;

$\therefore$ log.100·345 $= y_x = 2 +$ ·345[·0043424 − ·345(·0000209 + ·345 · ·0000001$\frac{1}{3}$)].

Operation.

```
     1⅓              43424
    ·345                72
    ----             -----|
     345             43352|
     115                  |345
    ----             -----|---
    ·460             13005|6*
 209·                 1734|1
 -------               216|7
 209·46             ------|-
    ·345            ·0014956
  ------            2·0000000
  62·7*            ----------
   8·4             2·0014956 = log.100·345.
   1·0
  -----
  72·1
```

What is the logarithm of 101·7906? Ans. 2·007709.

Required, the logarithms of 100·1, 100·2, 100·3, 100·4, 100·5, 100·6, 100·7, 100·8, 100·9.

When it is required, as in this example, to interpolate an arithmetical progression of tenths, form (280) may be advantageously modified, as follows. Suppose that we have found the logarithm corresponding to $x=\frac{r}{10}$ and wish the logarithm for $x=\frac{r+1}{10}$, we have (280)

$$y_{\frac{r}{10}}=y_0+C_1\cdot\frac{r}{10}-C_2\cdot\frac{r^2}{100}-C_3\cdot\frac{r^3}{1000},$$

$$y_{\frac{r+1}{10}}=y_0+C_1\cdot\frac{r+1}{10}-C_2\cdot\frac{r^2+2r+1}{100}-C_3\cdot\frac{r^3+3r^2+3r+1}{1000};$$

$$\therefore\quad y_{\frac{r+1}{10}}-y_{\frac{r}{10}}=C_1\cdot\frac{1}{10}-C_2\cdot\frac{2r+1}{100}-C_3\cdot\frac{3r^2+3r+1}{1000},$$

$$\text{or }\ y_{\frac{r+1}{10}}=y_{\frac{r}{10}}+\frac{C_1}{10}-\frac{C_2}{100}-\frac{C_3}{1000}-r\left[\frac{2C_2}{100}+(r+1)\cdot\frac{3C_3}{1000}\right].\qquad(281)$$

To adapt this form to the computation of the tenths from 100 to 101, we have

* Shortened multiplication. Rule?

$$\begin{array}{rr|l}
C_1 = & 43424, & \therefore \quad \dfrac{C_1}{10} = 4342{\cdot}4, \\
C_2 = & 209, & -\dfrac{C_2}{100} = -\ 2{\cdot}09, \\
C_3 = & 1\,; & -\dfrac{C_3}{1000} = -{\cdot}001\,;
\end{array}$$

$$\therefore \qquad y_{\frac{r+1}{10}} = y_{\frac{r}{10}} + 4340{\cdot}309 - r[4{\cdot}18 + (r+1)\cdot{\cdot}003].$$

Making $r = 0$, we have, log. 100·1 $= y_{\frac{1}{10}} =$ log. 100 + ·0004340

$= 2{\cdot}0004340\,;$

$r = 1$, gives	4340·309			$r = 3$, gives	13008
	4·186				4340
		$r = 2$, gives	8676		12
	4336·123		4340	100·4)	2·0017336
	8676·432		8		16
∴ log. 100·2 =	2·0008676 ;	log. 100·3 =	2·0013008 ;	100·5)	2·0021660.

Adapt (281) to interpolate between 101 and 102, and compute the logarithms of 101·1, 101·2, ..., 101·9.

For returning from the logarithm to its number, we have (280)

$$\frac{y_x - y_0}{C_1 - x(C_2 + x\,C_3)} = x, \tag{282}$$

where, it is to be observed, that C_1 is a very near trial divisor, though too great. We may therefore find an approximate value of x by dividing by C_1, and then, having perfected the divisor, repeat the operation.

Given the logarithm 2·0014956 to find the corresponding number.

Operation.

2·0014956	·344	43|3|5|2)14956(·345
2·0000000	$C_3 =\ 1\tfrac{1}{7}$	13006
$C_1 =$ 4|3|4|24*)1495|6(·344	·115	1950
1303	209·459	1734
192	·344	216
175	62|8	216
17	8|4	
	|8	∴ No. 100·345
	72	

* Shortened division.

If we had divided simply by $D_1 = 43214$, we should have found '346, which is near the truth, and the approximation will be still nearer as we ascend above 100; so that ordinarily it will be sufficient to diminish the given logarithm by the tabular logarithm next below it, and divide this difference by the difference of the tabular logarithms above and below. And when greater accuracy is required, the third difference may generally be neglected, whereby (280) and (282) will be reduced to

$$y_x = y_0 + x(D_1 + \tfrac{1}{2}D_2 - x \cdot \tfrac{1}{2}D_2), \quad \frac{y_x - y_0}{D_1 + \frac{1}{2}D_2 - x \cdot \frac{1}{2}D_2} = x. \qquad (283)$$

Further it will be sufficient to correct the divisor by the first digit of the quotient, or the nearest to it, which may be found by inspection. The operation above becomes

$$\begin{array}{r|l}
D_1 = 4\;3\;2\;1\;4 & \\
\tfrac{1}{2}D_2 = \quad 2\;1\;3 & \\
\hline
4\;3\;4\;2\;7 & 14956('3449\tfrac{1}{4} \\
6\;4 & 13009 \\
\hline
4|3|3|6|3 & \;1947 \\
 & \;1734 \\
\cline{2-2}
 & \quad 213 \\
 & \quad 173 \\
\cline{2-2}
 & \quad\; 40 \\
 & \quad\; 39 \\
\cline{2-2}
 & \quad\;\; 1
\end{array}$$

The *Characteristic*, or integral part of the logarithm, is not usually inserted in the tables, since it can readily be determined from the relation

$$10^y = x;$$

which gives

for $\quad y = 0,\ 1,\ 2,\ 3, \ldots, -1, -2, -3, \ldots;$
$\qquad x = 1,\ 10,\ 100,\ 1000, \ldots,\ '1,\ '01,\ '001, \ldots;$

Hence the characteristic is always indicated by the distance of the first significant figure from the place of units; + if to the left, − if to the right. (284)

Thus the characteristic, or integral part of the logarithm answer-

ing to the number 365 is 2; because 3 is two places distant from the units. Therefore, bearing (267) in mind, we have

$$\log. 365 = 2{\cdot}56229, \qquad \text{[See tables].}$$
$$\log. 36{\cdot}5 = \log. \tfrac{365}{10} = \log. 365 - \log. 10 = 2{\cdot}56229 - 1, = 1{\cdot}56229,$$
$$\log. 3{\cdot}65 = \log. \tfrac{365}{100} = (2-2){\cdot}56229 = 0{\cdot}56229,$$
$$\log. {\cdot}365 = \log. \tfrac{365}{1000} = (2-3){\cdot}56229 = \bar{1}{\cdot}5\overset{+}{6}229,$$
$$\log. {\cdot}0365 = \log. \tfrac{365}{10000} = (2-4){\cdot}56229 = \bar{2}{\cdot}56229; \ \&c., \&c.$$

It will be seen from these examples that the logarithm of a number in part or wholly decimal, is to be found in the same way as if it were integral, except the characteristic which is determined by the rule above and may be either plus or minus, while the tabular part of the logarithm is always plus.

EXERCISES.

1°. *Multiplication* (264).

	Operation.	Multiply	by
Multiply 465	2·66745	366	15·7,
by 37·7.	1·57634	90109	·035,
		·007	·034.
Ans. 17530.	4·24379		

2°. *Division* (267).

	Operation.	Divide	by
Divide ·054	$\bar{2}{\cdot}73239$	375·13	4095,
by 1·75.	0·24304	·0005	·00789.
Ans. ·030857.	$\bar{2}{\cdot}48935$		

3°. *Involution* (265).

	Operation.	
Cube 17·356.	1·23945	Square 139·75.
	3	$(1{\cdot}0399)^4 = ?$ $\left(\frac{113}{353}\right)^3 = ?$
Ans. 5228·2.	3·71835	

4°. *Evolution* (266).

	Operation.	
Find $\sqrt[5]{2}$	5|·3010300	Find $\sqrt{365}$, $\sqrt[3]{{\cdot}001}$,
Ans. 1·148702	·0602060	$(75{\cdot}00005)^{\frac{2}{3}}$, $\sqrt{{\cdot}2}$,
		$\sqrt[3]{{\cdot}2}$, $\sqrt{{\cdot}02}$, $\sqrt{{\cdot}002}$.

Note. An operation should be so conducted as to restrict the

minus sign to the characteristic. Thus, in finding the square root of '2, we have to divide $\bar{1}$'3010300 by 2, which is readily performed by observing that $\bar{1}=\bar{2}+\overset{+}{1}$, we have

$$\bar{1}\text{'}3010300 : 2 = \bar{1}\text{'}6505150;$$

so $$\bar{3}\text{'}3010300 : 2 = (\bar{4}+\overset{+}{1}\text{'}3010300) : 2 = \bar{2}\text{'}6505150.$$

5°. *Rule of Three.* Find a fourth proportional to the

Numbers,	*Operation.*	
13'79,	1'13956	
99'367,	1'99724	
720'25.	2'85748	1'005 : '00356 : : 79'099 : ?
Ans. 5189'9.	3'71516	$\frac{2057106}{3379071} : \frac{3657}{7112} : : \frac{5892167}{1\text{'}000009} : ?$

6°. *Exponential Equations—Compound Interest.*

If a denote the amount of one dollar for one year at compound interest, then

$$aa = a^2 \text{ will} = \text{the amount of \$1 for 2 years,}$$
$$a^2a = a^3 = \text{amount of \$1 for 3 years,}$$
$$a^3a = a^4 = \text{amount of \$1 for 4 years, ...,}$$
$$a^t = \text{amount of \$1 for } t \text{ years;}$$

$\therefore$ $Pa^t =$ amount of \$$P$ for t years $= A$, is the equation for compound interest. Taking the logarithms of both sides, we have

$$\log.\ (Pa^t) = \log.\ A,$$

or $$\log.\ P + \log.\ (a^t) = \log.\ A,$$

or $$\log.\ P + t\log.\ a = \log.\ A,$$

the logarithmic equation for compound interest. It furnishes, also, an example of the solution of an *Exponential Equation*, since the exponent t may be the unknown quantity, and we have

$$t = \frac{\log.\ A - \log.\ P}{\log.\ a}.$$

Application. In how long time will any sum of money double at compound interest, at 7 per centum?

PROPOSITION V.

It is required to develope an exponential function in terms of the exponent;

$$y = a^x, \tag{285}$$

in terms of x. Attributing to y and x the increments k and h, we have

$$y + k = a^{x+h} = a^x \cdot a^h,$$

$$\therefore \quad k = a^x(a^h - 1) = a^x[(1+b)^h - 1], \quad [\text{putting } a = 1 + b],$$

$$\text{or} \quad k = a^x[1 + hb + h(h-1)\frac{b^2}{2} + \ldots - 1];$$

$$\therefore \quad \frac{k}{h} = a^x[b + (h-1) \cdot \frac{b^2}{2} + (h-1)(h-2) \cdot \frac{b^3}{2 \cdot 3} + \ldots];$$

$$\therefore \quad y'_{y=a^x} = \left[\frac{k}{h}\right] = a^x[b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} +, -, \ldots] = Aa^x, \quad (286)$$

putting
$$A = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} +, -, \ldots$$

$$= (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} +, -, \ldots = \frac{1}{M}. \quad (287)$$

[See (268), (269).] That is, the derivative of an exponential is found by multiplying the function itself by a constant quantity which is the reciprocal of the modulus of a system of logarithms to the same base.

Taking the derivative of (286), or the second derivative of (285), we find

$$y'' = (y')' = A \cdot Aa^x = A^2a^x; \text{ so } y''' = A^3a^x, \; y^{iv} = A^4 a^x, \ldots;$$

and making $x = 0$, we have

$$y_{x=0} = a^0 = 1, \; y'_{x=0} = A, \; y''_{x=0} = A^2, \; y'''_{x=0} = A^3, \ldots,$$

whence we are assured, precisely as in the demonstration of the Binomial Theorem, that the function $y = a^x$, is expansible in terms affected by integral additive power of x and by no others, so that the form

$$y = a^x = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n + \ldots,$$

is possible, and no other. Therefore taking the derivatives, we obtain

$$y' = Aa^x = C_1 \cdot 1 + C_2 \cdot 2x + C_3 \cdot 3x^2 + C_4 \cdot 4x^3 + \ldots$$
$$+ C_n \cdot nx^{n-1} + \ldots,$$

$$y'' = A^2a^x = C_2 \cdot 2 \cdot 1 + C_3 \cdot 3 \cdot 2x + C_4 \cdot 4 \cdot 3x^2 + \ldots$$
$$+ C_n \cdot n(n-1)\, x^{n-2} + \ldots,$$

$$y''' = A^3a^x = C_3 \cdot 3 \cdot 2 \cdot 1 + C_4 \cdot 4 \cdot 3 \cdot 2x + \ldots$$
$$+ C_n \cdot n(n-1)(n-2)x^{n-3} + \ldots, \text{ \&c., \&c., \&c.,}$$

$$y^{(n)} = A^na^x = C_n \cdot n(n-1)(n-2) \ldots \cdot 3 \cdot 2 \cdot 1$$
$$+ C_{n+1} \cdot (n+1)n(n-1) \ldots \cdot 3 \cdot 2x + \ldots;$$

and, making $x = 0$, there results

$$C_0 = 1,\ C_1 = A,\ C_2 = \frac{A^2}{1 \cdot 2},\ C_3 = \frac{A^3}{1 \cdot 2 \cdot 3}, \ldots$$

$$C_n = \frac{A^n}{1 \cdot 2 \cdot 3 \ldots \cdot n}, \ldots,$$

which substituted above gives

THE EXPONENTIAL THEOREM.

$$y = a^x = 1 + Ax + \frac{A^2x^2}{1 \cdot 2} + \frac{A^3x^3}{1 \cdot 2 \cdot 3} + \ldots + \frac{A^nx^n}{1 \cdot 2 \cdot 3 \ldots \cdot n} + \ldots, \qquad (288)$$

$$\text{or } y = 1 + Ly + \frac{A^2(Ly)^2}{1 \cdot 2} + \frac{A^3(Ly)^3}{1 \cdot 2 \cdot 3} + \ldots.$$

This last may be called the *Antilogarithmic Series.*

In (287) making $M = 1$, A also $= 1$ and a becomes the base of the Napierian system, which it is customary to denote by e. Conforming (288) to these conditions, we have

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \ldots \qquad (289)$$

in which, making $x = 1$, there results

$$\left.\begin{matrix} \textit{Napierian} \\ \textit{Base} \end{matrix}\right\} = e = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots = 2{\cdot}7182818. \qquad (290)$$

Again, as A and x are both arbitrary, we may put $Ax = 1$, or $x = \frac{1}{A}$; whence (288) becomes

$$a^{\frac{1}{A}} = 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots = e;$$

$$\therefore\ e^A = a, \text{ or } A = \frac{La}{Le} = \frac{\log.\ a}{\log.\ e} = \frac{la}{le} = \frac{la}{1} = la, \qquad (291)$$

is the relation between the constant A and the base a of any system.

SECTION THIRD.

General Laws relating to the development of Functions depending on a single Variable.

PROPOSITION I.

To find the ratio of the increment of any continuous function to that of its variable.

The Theory, or Arithmetic of Functions, like the rules of algebra, constitutes a distinct science; for it is capable of development independently of any particular or denominate quantity, being equally applicable to problems pertaining to numbers, lines, forces, &c. But, as an infinite variety of curves may be drawn upon a plane, differing from each other in their laws of curvature in every possible way, any continuous function, y, depending on a single variable, x, may be represented by the ordinate of a continuous (*fig.* 48.) curve PP_2, that is, by a curve that bends everywhere by insensible degrees, of which the variable x will be the abscissa; and there will be an advantage in such a representation, as the understanding will be aided by the geometrical figure.

We have (234)

$$\frac{k}{h} = \frac{y}{X_y X_s} = \frac{y}{X_y X_t} + z,$$

putting $$z = \frac{y}{X_y X_s} - \frac{y}{X_y X_t} = \frac{y(X_y X_t - X_y X_s)}{(X_y X_s)\cdot(X_y X_t)},$$

from which it appears that z diminishes continuously with h and vanishes at the same time; since it is obvious that, as P_2 approaches to coincidence with P by a continuous curve, X_s must also approximate continuously to a coincidence with X_{t}, when

z becomes $$= \frac{y(X_y X_t - X_y X_t)}{(X_y X_t)\cdot(X_y X_t)} = \frac{y\cdot 0}{(X_y X_t)^2} = 0.$$

Therefore observing that (234_2)

$$\frac{y}{X_y X_t} = \left[\frac{k}{h}\right] = y'_{y=fx}, \left[y'_{y=fx}, \begin{matrix}\text{the derivative of } y \text{ when} \\ y \text{ is a function of } x.\end{matrix}\right]$$

we have $$\frac{k}{h} = \left[\frac{k}{h}\right] + z = y'_{y=fx} + z\,; \text{ i. e.,} \qquad (292)$$

The ratio of the increment of any continuous function to that of its variable, differs from the derivative of the same variable, by a quantity which diminishes continuously with the increment of the variable, so as to vanish at the same instant.

PROPOSITION II.

To find the derivative of a polynomial, the terms of which are continuous functions of the same variable.

Let $y = u + v + \ldots + constant$, be the polynomial, where y, u, v, ..., are functions of x, viz.,

$$y = fx, \; u = f_1 x, \; v = f_2 x, \ldots;$$

and let the corresponding increments be k, i, j, ..., h; we have

$$y + k = (u + i) + (v + j) + \ldots + constant;$$

$\therefore$ $$k = i + j + \ldots,$$

and $$\frac{k}{h} = \frac{i}{h} + \frac{j}{h} + \ldots,$$

or (292) $$y'_{y=fx} + z = (u'_{u=f_1x} + z_1) + (v'_{v=f_2x} + z_2) + \ldots,$$

and, z, z_1, z_2, ... becoming $= 0$ when $h = 0$,

we have $$y'_{y=fx} = u'_{u=f_1x} + v'_{v=f_2x} + \ldots, \tag{293}$$

or $$f'x = (f_1 x + f_2 x + f_3 x + \ldots + constant)'$$
$$= f_1'x + f_2'x + f_3'x + \ldots; \text{ i. e.,}$$

The derivative of a polynomial, consisting of continuous functions of the same variable, may be found by forming the algebraical sum of the derivatives of its several terms.

Illustration. The student has already had particular examples of this proposition, as in (247), (248), where x^a, x^b, x^c, ... are functions of x.

Cor. If the functions f_1, f_2, f_3, ... be all the same and a in number, (293) reduces to

$$(af_1 x + constant)' = af_1'x;$$

hence, from $$f_1 x = \frac{m}{n} f_2 x,$$

or from $$nf_1 x = mf_2 x,$$

we have $$nf_1'x = mf_2'x,$$

or $$\left(\frac{m}{n} f_2 x\right)' = f_1'x = \frac{m}{n} f_2'x; \text{ i. e.} \tag{294}$$

The derivative of a multiple or submultiple function, is equal to the same multiple or submultiple of its derivative.

Illustration. The derivative of x^n is mx^{n-1}, of Ax^n is $A \cdot mx^{n-1}$.

PROPOSITION III.

To find a derivative by the aid of intermediate functions.

When y is a function of u, and u a function of x, the corresponding increments being k, i and h, (292) gives

for $$y = f_1 u, \; \frac{k}{i} = y'_{y-f_1u} + z_1,$$

for $$u = f_2 x, \; \frac{i}{h} = u'_{u-f_2x} + z_2;$$

$$\therefore \; \frac{k}{h} = \frac{k}{i} \cdot \frac{i}{h} = (y'_{y-f_1u}) \cdot (u'_{u-f_2x}) + (y'_{y-f_1u}) \cdot z_2$$
$$+ (u'_{u-f_2x}) \cdot z_1 + z_1 z_2;$$
$$\therefore \; y'_{y-fx} = (y'_{y-f_1u}) \cdot (u'_{u-f_2x}),$$

making h, and, consequently, k, z_1, z_2, $= 0$.

So from $$y = f_1 u, \; u = f_2 v, \; v = f_3 x,$$
we find $$y'_{y-fx} = (y'_{y-f_1u}) \cdot (u'_{u-f_2v}) \cdot (v'_{v-f_3x});$$
and generally $$y_x' = y_s' \cdot s_t' \ldots \cdot w_x'; \text{ i. e.,} \qquad (295)$$

When several variables are successively functions of each other, the continued product of their derivatives, taken in the same order of succession, will be the derivative of the first, regarded as a function of the last.

Illustration. See Proposition II., Section First, $y = f_1 z = Az^n$, $z = f_2 x = x^{\frac{1}{n}}$.

Cor. The derivatives of converse functions are reciprocals of each other.* (296)

For, let y and x be functions of each other, or
$$y = fx \text{ and } x = \phi y,$$
then we have $$(y'_{y-fx}) \cdot (x'_{x-\phi y}) = y'_{y-y} = 1,$$
or $$x'_{x-\phi y} = \frac{1}{y'_{y-fx}}.$$

Illustration. In (285) let x and y change places, and compare (286) with (268).

---

* The chord and arc of a circle are conversely functions of each other.

PROPOSITION IV.

To find the derivative of a product, the factors of which are continuous functions of the same variable.

Given $y = fx$; $y = uv$, $u = f_1 x$, $v = f_2 x$; to find $y'_{y=fx}$.

We have $$y + k = (u+i)(v+j) = y + vi + uj + ij,$$

$$\therefore \quad k = vi + uj + ij;$$

but $$\frac{i}{h} = u'_{u=f_1 x} + z_1 \text{ or } i = (u'_x + z_1)h,$$

and $$\frac{j}{h} = v'_{v=f_2 x} + z_2, \text{ or } j = (v'_x + z_2)h;$$

$$\therefore \quad k = v(u'_x + z_1)h + u(v'_x + z_2)h + (u'_x + z_1)(v_x + z_2)h^2;$$

$$\therefore \quad \frac{k}{h} = v(u'_x + z_1) + u(v'_x + z_2) + (u'_x + z_1)(v'_x + z_2)h,$$

and $$y'_x = \left[\frac{k}{h}\right] = (uv)' = vu'_x + uv'_x,$$

or $$\frac{(uv)'}{uv} = \frac{u'}{u} + \frac{v'}{v}.$$

In the same way, if v be $= st$, continuous functions of x, we have

$$\frac{v'}{v} = \frac{(st)'}{st} = \frac{s'}{s} + \frac{t'}{t},$$

which, substituted above, gives,

$$\frac{(stu)'}{stu} = \frac{s'}{s} + \frac{t'}{t} + \frac{u'}{u},$$

and generally $$\frac{(stu\ldots)'}{stu\ldots} = \frac{s'}{s} + \frac{t'}{t} + \frac{u'}{u} + \ldots, \qquad (297)$$

or $$\frac{(f_1 x f_2 x f_3 x \ldots)'}{f_1 x f_2 x f_3 x \ldots} = \frac{f_1' x}{f_1 x} + \frac{f_2' x}{f_2 x} + \frac{f_3' x}{f_3 x} + \ldots,$$

or $$\frac{(f_1 \cdot f_2 \cdot f_3 \ldots)'}{f_1 \cdot f_2 \cdot f_3 \ldots} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \frac{f_3'}{f_3} + \ldots,$$

omitting the variable x, and employing only the symbols of operation, $f_1, f_2, f_3, \ldots$, which may be done, since they are equally applicable to any quantity which may be made the independent variable; thus, instead of $\sqrt[2]{x} \cdot \sqrt[3]{x} = \sqrt[5]{x}$, we may write $\sqrt[2]{} \cdot \sqrt[3]{} = \sqrt[5]{}$, as a general rule. We may enunciate (297):

The derivative of a product of continuous functions, divided by the product itself, is equal to the sum of the derivatives of the functions divided by these functions severally.

PROPOSITION V.

To find the derivative of any real power of a continuous function.

If in (297) we make the functions s, t, u, ..., all the same and n in number, we find

$$\frac{(u^n)'}{u^n} = n \cdot \frac{u'}{u}, \text{ or } (u^n)' = nu^{n-1} \cdot u'. \qquad (a)$$

If $\qquad y = u^{-r}$, u being $= f_1 x$,

we have $\qquad yu^r = u^{-r} \cdot u^r = u^0 = 1$,

$\therefore$ (297) $\qquad \dfrac{(yu^r)'}{yu^r} = \dfrac{y'}{y} + \dfrac{(u^r)'}{u^r} = \dfrac{(1)'}{1} = \dfrac{0}{1} = 0;$

$\therefore$ (a) $\qquad y' = (u^{-r})' = -y \cdot \dfrac{ru^{r-1} \cdot u'}{u^r} = -u^{-r} \cdot \dfrac{ru^{r-1} \cdot u'}{u^r}$

$$= -ru^{-r-1} \cdot u'. \qquad (b)$$

If $\qquad y = fx = u^{\frac{m}{n}}$, u being $= f_1 x$;

we have $\qquad y = v^m$, putting $v = u^{\frac{1}{n}}$, or $u = v^n$;

$\therefore$ (a) or (b), $\qquad y_v' = mv^{m-1}$, and $u_v' = nv^{n-1}$,

or (296), $v_u' = \dfrac{1}{u_v'} = \dfrac{1}{nv^{n-1}}$;

$\therefore$ (295), $\qquad y_x' = y_v' \cdot v_u' \cdot u_x' = mv^{m-1} \cdot \dfrac{1}{nv^{n-1}} \cdot u_x'$,

or $\qquad y_x' = \dfrac{m}{n} u^{\frac{m}{n}-1} \cdot u_x'. \qquad (c)$

From a comparison of (a), (b), (c), it appears that the same rule holds for the derivative of a power of a function, whatever real quantity the exponent may be; and we may write

$$[(fx)^n]' = n(fx)^{n-1}(fx)', \qquad (298)$$

or $\qquad [(f)^n]' = n(f)^{n-1}(f)'$; i. e.,

The rule found in the first section for the derivative of any power of a variable, holds for the power of a function.

Indeed (298) embraces (245); for if $fx = x$, then $(fx)' = x' = 1$, and (298) reduces to $(x^n)' = nx^{n-1}$.

PROPOSITION VI.

To find the derivative of a fraction, the terms of which are continuous functions of the same variable.

From (297) we have

$$(f_1 \cdot f_2)' = f_1 \cdot f_2\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2}\right) = f_2 \cdot f_1' + f_1 \cdot f_2' ;$$

$\therefore$ putting $\quad f_1 = f_n, f_2 = \left(\frac{1}{f_d}\right) = (f_d)^{-1},$

$$[f_n \cdot (f_d)^{-1}]' = (f_d)^{-1} \cdot f_n' + f_n \cdot [(f_d)^{-1}]',$$

but (298), $\quad [(f_d)^{-1}]' = -1(f_d)^{-1-1} \cdot f_d' ;$

$$\therefore \quad \left(\frac{f_n}{f_d}\right)' = \frac{f_n'}{f_d} - \frac{f_n \cdot f_d'}{(f_d)^2} = \frac{f_d \cdot f_n' - f_n \cdot f_d'}{(f_d)^2}; \text{ i. e.,} \qquad (299)$$

The derivative of a fraction whose terms are continuous functions of the same variable, is equal to the denominator multiplied into the derivative of the numerator minus, the numerator multiplied into the derivative of the denominator, divided by the square of the denominator.

$$\therefore \quad \left(\frac{f_n}{f_d}\right)' : \left(\frac{f_n}{f_d}\right) = \frac{f_n'}{f_n} - \frac{'_d}{f_d}. \qquad (299)$$

PROPOSITION VII.

In finding the derivative of any continuous function (300) y = fx, *we may replace the increments* k, h, *one or both, by such quantities,* k_1, h_1, *as are separately functions of* k *and* h, *and such that the final or vanishing ratios* $k_1 : k$, $h_1 : h$, *become that of unity.*

For, by hypothesis, we have

$\frac{k_1}{k} = 1 + z_1$, z_1 being such as to reduce to zero when k and k_1 become $= 0$;

or $\quad k_1 = (1 + z_1)k$, so $h_1 = (1 + z_2)h$;

$$\therefore 292 \quad \frac{k_1}{h_1} = \frac{1 + z_1}{1 + z_2} \cdot \frac{k}{h} = \frac{1 + z_1}{1 + z_2}(y'_{y=fx} + z) ;$$

but when h becomes $= 0$, $\frac{1 + z_1}{1 + z_2}$ reduces to $\frac{1 + 0}{1 + 0} = 1$, and we have

$$\left[\frac{k_1}{h_1}\right] = \left[\frac{k}{h}\right] = y_{y=fx}. \quad \text{Q. E. D.}$$

PROPOSITION VIII.

To find the expansion of a continuous function, such that its successive derivatives all become finite when the independent variable reduces to zero.

Let $y = fx$ be the function. It follows, from a process of reasoning precisely like that employed in the demonstration of the Binomial Theorem, that no other than integral additive powers of x can enter into the expansion; and it only remains (and is sufficient) to see if the assumption

$$y = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots,$$

is possible, or, what amounts to the same thing, if the coëfficients A_0, A_1, A_2, A_3, ..., are determinable, and, therefore, real.

Taking the successively derived functions, we find

$$\begin{aligned}
y' &= A_1 \cdot 1 + A_2 \cdot 2x + A_3 \cdot 3x^2 + \dots, \\
y'' &= A_2 \cdot 2 \cdot 1 + A_3 \cdot 3 \cdot 2x + A_4 \cdot 4 \cdot 3x^2 + \dots, \\
y''' &= A_3 \cdot 3 \cdot 2 \cdot 1 + A_4 \cdot 4 \cdot 3 \cdot 2x + A_5 \cdot 5 \cdot 4 \cdot 3x^2 + \dots, \\
y^{\text{iv}} &= A_4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + A_5 \cdot 5 \cdot 4 \cdot 3 \cdot 2x + \dots, \\
&\text{\&c.,} \qquad \text{\&c.,} \qquad \text{\&c.,} \qquad \text{\&c.,}
\end{aligned}$$

Now, if in the above equations we make $x = 0$, and indicate the corresponding finite values of

$$y,\ y',\ y'',\ y''',\ y^{\text{iv}},\ \dots,$$

by

$$y_0,\ y_0',\ y_0'',\ y_0''',\ y_0^{\text{iv}},\ \dots,$$

there results $y_0 = A_0$, $y_0' = A_1 \cdot 1$, $y_0'' = A_2 \cdot 2 \cdot 1$, $y_0''' = A_3 \cdot 3 \cdot 2 \cdot 1$, ...;

$$\therefore \quad y = y_0 + y_0' \cdot \frac{x}{1} + y_0'' \cdot \frac{x^2}{1 \cdot 2} + y_0''' \cdot \frac{x^3}{1 \cdot 2 \cdot 3}$$

$$+ y_0^{\text{iv}} \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dots. \qquad (301)$$

This is essentially *Maclaurin's Theorem*, and is very serviceable in expansions, being more general than the Binomial, which it becomes simply by putting $y = (a + x)^n$.

EXERCISES.

1°. $(\log.\ x^2)' = \frac{M}{x^2} \cdot 2x = \frac{2M}{x}$; $\therefore (lx^2)' = \frac{2}{x}$. [(295), (268), (245)].

2°. $[(\log.\ x)^2]' = \frac{2M \log.\ x}{x}$; $\therefore [(lx)^2]' = \frac{2lx}{x}$.

3°. $(\log.\ x^n)' = \frac{Mn}{x}$; $\therefore (lx^n)' = \frac{n}{x}$.

4°. $[(\log.\ x)^n]' = \frac{Mn(\log.\ x)^{n-1}}{x}$.

5°. $[\log.\ (ax^n + b)]' = \frac{Manx^{n-1}}{ax^n + b}$.

6°. $[(alx + b)^n]' = \frac{an(alx + b)^{n-1}}{x}$.

7°. $(\log.\ a^x)' = \frac{M}{a^x} \cdot Aa^x = MA = M \cdot \frac{1}{M} = 1$. [(286, (296).]

8°. $[(lx)^x]' = \frac{(lx)^x}{x}$.

9°. $[l(fx)^n]' = \frac{n(fx)'}{fx}$. [(298)]

10°. $[l(a + bx + cx^2 + \ldots)^n]' = \frac{n(b + c \cdot 2x + d \cdot 3x^2 + \ldots)}{a + bx + cx^2 + dx^3 + \ldots}$ [(293)]

11°. $\left(\log.\ \frac{f_n}{f_d}\right)' = M\left(\frac{f_n'}{f_n} - \frac{f_d'}{f_d}\right)$. [(299)]

12°. $\left(\log.\ \frac{a+u}{a-u}\right)' = M \cdot \frac{2au'}{a^2 - u^2}$; $[u = fx]$.

13°. $\left[l\left(\frac{a+u}{a-u}\right)^n\right]' = \frac{2nau'}{a^2 - u^2}$. [[(298)]]

14°. $\left[l\left(\frac{f_n}{f_d}\right)^n\right]' = n\left[\frac{f_n'}{f_n} - \frac{f_d'}{f_d}\right]$.

BOOK SECOND.

PLANE TRIGONOMETRY.

SECTION FIRST.

Trigonometrical Analysis.

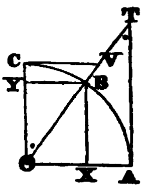


Fig. 52.

Construction. Describe the quadrant ABC; drop the perpendiculars BX, BY, upon the radii OA, OC; produce OB so as to intercept AT and CV, perpendiculars drawn through the extremities of OA, OC, in T and V; then:

Definition 1. The arcs AB, BC, are said to be *complementary* to each other. BC is the complement of AB and AB is the complement of BC; the arc $90° - a$ is the complement of a and a is the complement of $90° - a$; $45° + x$ and $45° - x$ are complementary arcs.

Def. 2. The perpendicular BX is called the *sine* of the arc AB. Hence the sine of an arc is the perpendicular let fall from one extremity of the arc upon the diameter passing through the other extremity of the same arc.

Def. 3. AT is the *tangent* of AB.

Def. 4. OT is the *secant* of AB.

Def. 5. BY (= OX) is the sine of BC or the cosine of AB.

Def. 6. CV is the tangent of BC or the *cotangent* of AB.

Def. 7. OV is the secant of BC or the *cosecant* of AB.

Def. 8. AX is the *verst sine* of AB.

Note. The abbreviations of the titles above, either with or without the period, are employed as symbols of the quantities themselves. Thus, if a denote any arc less than a quadrant and b any arc not greater than 45°, the above definitions give

$$\left.\begin{array}{l}\sin a = \cos(90^\circ - a),\ \cos a = \sin(90^\circ - a);\\ \sin(90^\circ - a) = \cos a,\ \cos(90^\circ - a) = \sin a;\\ \sin(45^\circ + b) = \cos(45^\circ - b),\ \cos(45^\circ + b) = \sin(45^\circ - b).\end{array}\right\} \quad (302)$$

$$\tan a = \cot(90^\circ - a),\ \cot a = \tan(90^\circ - a);\ \&c. \quad (303)$$

$$\sec a = \operatorname{cosec}(90^\circ - a),\ \operatorname{cosec} a = \sec(90^\circ - a);\ \&c. \quad (304)$$

PROPOSITION I.

The sum of the squares of the sine and cosine of an arc is equal to the square of the radius, or to unity, when the radius is taken for the unit of the trigonometrical lines. (305)

We have

$$OB^2 = BX^2 + OX^2 = BX^2 + BY^2, \quad (\textit{fig. } 52.)$$

or $$\sin^2 a + \cos^2 a = r^2 = 1,$$

when radius $r = 1$.

Cor. The sine is an increasing and the cosine is a decreasing function of the arc, or the sine BX increases from 0 to r as the arc increases from 0 to 90°, while the cosine OX decreases from r to 0 for the same increase of the arc. (306)

See (194) and observe that the sine BX is half the chord of double the arc AB.

PROPOSITION II.

The tangent of an arc is to the radius as the sine to the cosine; or, the tangent is equal to the sine divided by the cosine, if the radius be taken for unity. (307)

We have $$\frac{TA}{OA} = \frac{BX}{OX}, \text{ or } \frac{\tan a}{r} = \frac{\sin a}{\cos a}, \quad (\textit{fig. } 52.)$$

or $$\tan a = \frac{\sin a}{\cos a},\ r = 1.$$

PROPOSITION III.

The radius is a mean proportional between the tangent and the cotangent of an arc; or, the tangent and cotangent of an arc are reciprocals of each other when r = 1. (308)

For, by similar triangles, we have (*fig.* 52.)

$$AT : AO = CO : CV, \text{ or } \tan a : r = r : \cot a;$$

$$\therefore \quad \tan a \cot a = r^2, = 1, \text{ when } r = 1.$$

PROPOSITION IV.

The square of the secant is equal to the sum of the squares of the radius and the tangent. (309)

We have

$$OT^2 = OA^2 + AT^2, \text{ or } \sec^2 a = r^2 + \tan^2 a. \qquad (fig. 52.)$$

The student may obtain other forms when wanted; as, for instance, the following:

The secant is to the tangent as radius to the sine. Also (310) secant $\times$ cosine $= r^2 = 1$.

PROPOSITION V.

An Incremental vanishing Arc *is to be regarded as a straight line perpendicular to the radius.* (311)

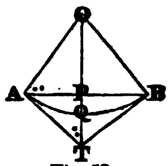


Fig. 53.

Let AB be the arc in question; draw the tangents AT, BT, intersecting in T, and join OT; then will the triangles AOT, BOT, be equal, and OT will bisect the chord and arc in P and Q and be perpendicular to AB. From the similar triangles TAP, TOA, we have

$$\frac{AT}{AP} = \frac{OT}{OA} = \frac{OQ + QT}{OQ} = 1 + \frac{QT}{OQ};$$

but $QT = OT - OQ = (OA^2 + AT^2)^{\frac{1}{2}} - OA$, which reduces to $[QT] = (OA^2 - 0^2)^{\frac{1}{2}} - OA = 0$, when the arc AQB becomes $= 0$, since then $AQ = \frac{1}{2}AQB = 0$ and $AT = \tan AQ = 0$; therefore the ultimate ratio of the vanishing quantities [AT], [AP], becomes

$$\left[\frac{AT}{AP}\right] = 1 + \left[\frac{QT}{OQ}\right] = 1 + \frac{0}{OQ} = 1, \text{ that of unity;}$$

and $\therefore$
$$\left[\frac{AT + TB}{AB}\right] = \left[\frac{2AT}{2AP}\right] = 1,$$

But (113) the arc AQB is greater than the chord AB, and less than the broken line ATB; $\therefore$ the quotient $\frac{AQB}{AB}$ is greater than $\frac{AB}{AB}$ or unity, and less than $\frac{ATB}{AB}$, which also becomes $= 1$, when the arc $AQB = 0$;

$\therefore$ the ratio $\left[\frac{\text{AQB}}{\text{AB}}\right]$ of the vanishing arc [AQB] to its vanishing chord [AB] cannot be less than one, and cannot be greater than one, and therefore must be $=1$; which proves the proposition (300), observing that the arc is perpendicular to the radius (177).

Scholium. It is not stated that the vanishing arc, when employed as an increment, merely *may* be regarded as a straight line perpendicular to the radius, but it is proved that it *must* be so regarded. On the other hand, it is to be observed that we do not affirm that the arc will ever actually become a straight line, or that it will not always exceed its chord in length, but that, for the purpose pointed out, it must be so regarded, in order to reduce it to zero, and thereby to eliminate it from the function under investigation.

PROPOSITION VI.

To find the derivatives of the sine and cosine regarded as functions of the arc.

Denoting the arc by x, the sine by y, and the cosine by z, the functions may be represented by

$$y=fx,\ z=\phi x,$$

which are the same as

$$y=\sin x,\ z=\cos x.$$

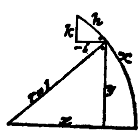


Fig. 54.

Attributing to x the incremental vanishing arc h, (311), and to y, z, the corresponding increments, $k, -i$, (306), the similar triangles, whose homologous sides, taken in order, are $k, h, -i$; $z, r=1, y$, give us

$$y'_{y=fx}=\left[\frac{k}{h}\right]=\frac{z}{r}=z;\ \text{i. e.,}$$

The derivative of the sine regarded as a function of its arc, is equal to the cosine, radius being unity. (312)

Again we have $\quad z'_{z=\phi x}=\left[\frac{-i}{h}\right]=\frac{-y}{r}=-y$; i. e.,

The derivative of the cosine regarded as a function of its arc, is equal to the sine taken minus. (313)

Proceeding to the 2d, 3d, 4th, &c., derivatives, observing that (294) gives $(-f)'=-(f')$ when $\frac{m}{n}=-1$, we find, (312), (313),

$y_x = \sin x,\ y_x' = +z_x = \cos x,\ y_x'' = z_x' = -y_x,\ y_x''' = z_x'' = (-y_x)'$
$= -y_x' = -z_x,\ y_x^{iv} = z_x''' = -z_x' = +y_x,\ y_x^{v} = z_x^{iv} = y_x'$
$= +z_x$, &c., &c., &c.; i. e.,

Cor. The sine and its derivatives are alternately *sine, cosine; sine, cosine;* ..., in which the algebraical signs alternate in pairs, $+, +; -, -; +, +; -, -;$ ..., and the cosine and its derivatives are alternately *cosine, sine; cosine, sine;* ..., in which the signs alternate in pairs, also alternating, $+, -; -, +; +, -; -, +;$ .... (314)

PROPOSITION VII.

To develope the sine and cosine in terms of the arc.

Let $y = \sin(a+x)$ and $z = \text{cosin}(a+x)$; then are y and z continuous functions of the arc x, $y = Fx$, $z = F_2 x$, which it is required to determine.

In (314) substituting $a+x$ for x, we find

$$y = \sin(a+x),\ y' = \cos(a+x),\ y'' = -\sin(a+x),$$
$$y''' = -\cos(a+x),\ y^{iv} = \sin(a+x),\ y^{v} = \cos(a+x),\ \dots,$$

which become

$$y_0 = \sin a,\ y'_0 = \cos a,\ y''_0 = -\sin a,\ y'''_0 = -\cos a,$$
$$y_0^{iv} = \sin a,\ y_0^{v} = \cos a,\ y_0^{vi} = -\sin a,\ \dots, \text{ when } x = 0;$$

$\therefore$ (301), $$\sin(a+x) = \sin a + \cos a \cdot \frac{x}{1} - \sin a \cdot \frac{x^2}{1\cdot 2} - \cos a \cdot \frac{x^3}{1\cdot 2\cdot 3} + \sin a \cdot \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \cos a \cdot \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} -, -, \dots,$$

or $$\sin(a+x) = \sin a\left(1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} +, -, \dots\right) + \cos a\left(\frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} +, -, \dots\right). \quad (315)$$

In like manner (314) we have

$$z_0 = \cos a,\ z_0' = -\sin a,\ z_0'' = -\cos a,\ z_0''' = \sin a,\ z_0^{iv} = \cos a,\ \dots;$$

$$\therefore \quad (301)\ \cos(a+x) = \cos a\left(1-\frac{x^2}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}\right. \qquad (316)$$

$$\left.-\frac{x^6}{1\cdot 2\ldots\cdot 6}+,-,\ldots\right)-\sin a\left(x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\ldots\cdot 5}\right.$$

$$\left.-\frac{x^7}{1\cdot 2\ldots\cdot 7}+,-,\ldots\right).$$

Making $a=0$, and observing that (306)

$$\sin(a_{a=0}) = \sin 0 = 0, \text{ and } \cos(a_{a=0}) = r = 1$$

there results, (315), (316),

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{x^7}{1\cdot 2\ldots\cdot 7}+,-,\ldots, \qquad (317)$$

and

$$\cos x = 1-\frac{x^2}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{x^6}{1\cdot 2\ldots\cdot 6}+,-,\ldots, \qquad (318)$$

and these are the developments required. They were discovered by Newton.*

If we change x into $-x$, (317) and (318) become (6), (6_6)

$$\sin(-x) = -x-\frac{-x^3}{1\cdot 2\cdot 3}+,-,\ldots = -\left(x-\frac{x^3}{1\cdot 2\cdot 3}+,-,\ldots\right)$$

$$= -\sin x,$$

$$\cos(-x) = 1-\frac{(-x)^2}{1\cdot 2}+,-,\ldots = 1-\frac{x^2}{1\cdot 2}+,-,\ldots = \cos x;\ \text{i. e.,}$$

Cor. The sine of an arc changes from + to − as the arc (319) itself changes from + to −, but the cosine remains still + while the arc passes through the value zero, which is in accordance with (180). Nothing, however, it is to be observed, has been demonstrated in regard to arcs greater than 90°, or $x >$ a quadrant.

PROPOSITION VIII.

It is required to express the sine and cosine of the sum and difference of two arcs in terms of the sines and cosines of the arcs themselves.

Changing x into $-x$, (315) and (316) become

$$\sin(a-x) = \sin a\left(1-\frac{x^2}{1\cdot 2}+,-,\ldots\right)-\cos a\left(x-\frac{x^3}{1\cdot 2\cdot 3}+,-,\ldots\right),$$

$$\cos(a-x) = \cos a\left(1-\frac{x^2}{1\cdot 2}+,\ldots\right)+\sin a\left(x-\frac{x^3}{1\cdot 2\cdot 3}+,-,\ldots\right);$$

---

* Lagrange, Leçons sur le Calcul des Fonctions.

with which and (315), (316), combining (317), (318), there results

$$\left.\begin{aligned} \sin(a+x) &= \sin a \cos x + \cos a \sin x, \\ \sin(a-x) &= \sin a \cos x - \cos a \sin x\,; \\ \cos(a+x) &= \cos a \cos x - \sin a \sin x, \\ \cos(a-x) &= \cos a \cos x + \sin a \sin x. \end{aligned}\right\} \quad (320)$$

These four forms are constantly recurring in trigonometrical analysis, and should therefore be committed to memory; they may be enunciated as follows:

I. *The sine of the sum of two arcs is equal to the sine of the first multiplied into the cosine of the second, plus the cosine of the first multiplied into the sine of the second.*

II. *The sine of the difference of two arcs is equal to the sine of the first multiplied into the cosine of the second, minus the cosine of the first multiplied into the sine of the second.*

III. *The cosine of the sum of two arcs is equal to the cosine of the first multiplied into the cosine of the second, minus the sine of the first multiplied into the sine of the second.*

IV. *The cosine of the difference of two arcs is equal to the cosine of the first multiplied into the cosine of the second, plus the sine of the first multiplied into the sine of the second.*

Consequences. Making $a = x$, we have (320)

Cor. 1. $\sin 2x = 2 \sin x \cos x,$ (321)

Cor. 2. $\cos 2x = \cos^2 x - \sin^2 x\,;$ (322)

but (305), $1 = \cos^2 x + \sin^2 x\,;$

which, combined with (322) and (321), gives

Cor. 3. $1 + \cos 2x = 2\cos^2 x,$ (323)

Cor. 4. $1 - \cos 2x = 2\sin^2 x\,;$ (324)

Cor. 5. $1 + \sin 2x = (\cos x + \sin x)^2,$ (325)

Cor. 6. $1 - \sin 2x = (\cos x - \sin x)^2\,;$ (326)

$\therefore$ *Cor.* 7. $(1 + \sin 2x)^{\frac{1}{2}} \pm (1 - \sin 2x)^{\frac{1}{2}} = 2\cos x,$ (327)

Cor. 8. $(1 + \sin 2x)^{\frac{1}{2}} \mp (1 - \sin 2x)^{\frac{1}{2}} = 2\sin x.$ (328)

What is the sine of a double arc? of half an arc? the cosine of a double arc? of half an arc? Enunciate (323), (324), (325), (326), (327), (328).

Adding and subtracting forms (320), and making $a + x = p$, $a - x = q$, and $\therefore\ a = \frac{1}{2}(p+q)$, $x = \frac{1}{2}(p-q)$, we have

Cor. 9. $\sin p + \sin q = 2\sin\frac{1}{2}(p+q) \cdot \cos\frac{1}{2}(p-q),$ (329)

Cor. 10. $\sin p - \sin q = 2\cos\frac{1}{2}(p+q) \cdot \sin\frac{1}{2}(p-q),$ (330)

Cor. 11. $\cos p + \cos q = 2\cos\frac{1}{2}(p+q) \cdot \cos\frac{1}{2}(p-q)$, (331)

Cor. 12. $\cos q + \cos p = 2\sin\frac{1}{2}(p+q) \cdot \sin\frac{1}{2}(p-q)$. (332)

These forms are useful in the application of logarithms, by converting sums and differences into products and quotients.

Dividing (329) by (330) we have (307), (308),

Cor. 13. $$\frac{\sin p + \sin q}{\sin p - \sin q} = \frac{\tan\frac{1}{2}(p+q)}{\tan\frac{1}{2}(p-q)}, \text{ i. e.,} \qquad (333)$$

The sum of the sines of two arcs is to their difference as the tangent of half their sum is to the tangent of half their difference.

By similar processes, other forms, occasionally useful, may be developed, as

Cor. 14. $$\frac{\cos p + \cos q}{\cos q - \cos p} = \cot\tfrac{1}{2}(p+q) \cdot \cot\tfrac{1}{2}(p-q), \qquad (334)$$

Cor. 15. $$\frac{\sin p \pm \sin q}{\cos p + \cos q} = \tan\tfrac{1}{2}(p \pm q), \qquad (335)$$

Cor. 16. $$\frac{\sin p \mp \sin q}{\cos q - \cos p} = \cot\tfrac{1}{2}(p \pm q). \qquad (336)$$

Making $a = 90°$, forms (320) become

$$\sin(90° + x) = \cos x,$$
$$\sin(90° - x) = \cos x\,;$$
$$\cos(90° + x) = -\sin x,$$
$$\cos(90° - x) = \sin x\,; \text{ i. e.,}$$

Cor. 17. The sines of supplementary arcs are equivalent, (337) being equal to the cosine of what one exceeds and the other falls short of 90°.

Cor. 18. The cosines of supplementary arcs are numerically equal, but have contrary algebraical signs. (338)

Arcs are *supplementary* when their sum amounts to 180°, as $(90° + x) + (90° - x) = 180°$.

In the above forms, making $x = 90°$, we have

$$\sin 180° = \sin(90° + 90°) = \cos 90° = 0,$$
$$\cos 180° = \cos(90° + 90°) = -\sin 90° = -1\,; \text{ i. e.,}$$

Cor. 19. The sine of 180° is $= 0$, and the cosine $= -1$. (339)

Cor. 20. $\sin(180° + x) = -\sin x$, (340)

Cor. 21. $\cos(180° + x) = -\cos x$; (341)

Cor. 22. $\sin 370° = -\sin 90° = -1$, (342)

Cor. 23. $\cos 370° = -\cos 90° = 0$; (343)

Cor. 24. $\sin(370° + x) = -\cos x$, (344)

Cor. 25. $\cos(370° + x) = +\sin x$. (345)

Scholium. The consequences evolved by these last forms (337) ... (345), are in accordance with the principle enounced in (180). Indeed, if we suppose the arc to increase from 0° to 360°, the sine will pass through the value 0 at 180° and again at 360°, while the cosine will reduce to zero at 90° and 270°; and it is obvious that the same correlation of values will be repeated in a 2d, 3d, &c., circumference. The algebraical sign of the tangent will be determined from the relation (307).

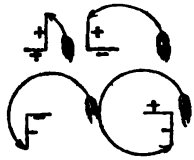


Fig. 55.

PROPOSITION IX.

It is required to develope the tangent and cotangent of the sum and difference of two arcs in terms of the tangents and cotangents of the arcs themselves.

Consulting (307) and (320), we obtain

$$\tan(a+b)=\frac{\sin(a+b)}{\cos(a+b)}+\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b-\sin a\sin b}$$

$$=\frac{\dfrac{\sin a}{\cos a}+\dfrac{\sin b}{\cos b}}{1-\dfrac{\sin a}{\cos a}\cdot\dfrac{\sin b}{\cos b}},$$

dividing numerator and denominator by $\cos a \cos b$;

$$\therefore\quad \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}, \tag{346}$$

is one of the relations sought.

So $$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}; \tag{347}$$

and (308), $$\cot(a\pm b)=\frac{1\mp\tan a\tan b}{\tan a\pm\tan b}=\frac{\cot a\cot b\mp 1}{\cot b\pm\cot a}. \tag{348}$$

Cor. 1. $$\tan 2a=\frac{2\tan a}{1-\tan^2 a}. \tag{349}$$

Cor. 2. $$\cot 2a=\frac{\cot^2 a-1}{2\cot a}. \tag{350}$$

Resolving equation (349) in reference to *tan a* and consulting (309), (310), we find

Cor. 3. $$\tan a = \frac{-1 + (1 + \tan^2 2a)^{\frac{1}{2}}}{\tan 2a} = \frac{-1 + \sec 2a}{\tan 2a} = \frac{1 - \cos a}{\sin 2a};\qquad (351)$$

Cor. 4. $$\operatorname{Cot} a = \cot 2a + (1 + \cot^2 2a)^{\frac{1}{2}} = \cot 2a + \operatorname{cosec} 2a = \frac{1 + \cos 2a}{\sin 2a}.\qquad (352)$$

Making $a = 45°$, $\cos 2a$ will $= 0$, $\sin 2a = 1$, and we shall have (351), (352),

Cor. 5. $$\tan 45° = 1 = \cot 45°.\qquad (353)$$
whence, making $a = 45°$ in (346), (347), (348), we find

Cor. 6. $$\tan(45° \pm b) = \frac{1 \pm \tan b}{1 \mp \tan b};\qquad (354)$$

Cor. 7. $$\cot(45° \pm b) = \frac{\cot b \mp 1}{\cot b \pm 1}\qquad (355)$$

Making $2a = 90° \pm u$ in (351), we get

Cor. 8. $$\tan(45° \pm \tfrac{1}{2}u) = \frac{1 \pm \sin u}{\cos u}\qquad (356)$$

Scholium. Other forms, serviceable in turning sums into products, and *vice versâ*, may be found; for example, if we make $p = 90°$ in (329), we get

$$\begin{aligned} 1 + \sin.\, q &= 2\sin(45° + \tfrac{1}{2}q)\cos(45° - \tfrac{1}{2}q) \\ &= 2\sin(45° + \tfrac{1}{2}q)\sin[90° - (45° - \tfrac{1}{2}q)] \\ &= 2\sin(45° + \tfrac{1}{2}q)\sin(45° + \tfrac{1}{2}q) \\ &= 2\sin^2(45° + \tfrac{1}{2}q).\qquad (357) \end{aligned}$$

So $$1 - \sin q = 2\cos^2(45° + \tfrac{1}{2}q) = 2\sin^2(45° - \tfrac{1}{2}q);\qquad (358)$$
$$1 + \cos p = 2\cos^2 \tfrac{1}{2}p,\ 1 - \cos p = 2\sin^2 \tfrac{1}{2}p.\qquad (359)$$

Combining (329), (330), (331), (332), and (321), we find
$$\sin^2 p - \sin^2 q = \cos^2 p - \cos^2 q = \sin(p+q)\sin(p-q),\qquad (360)$$
also $$\cos^2 p - \sin^2 q = \cos(p + q)\cos(p - q).\qquad (361)$$

The student will also find

$$\frac{\sin(a + b)}{\sin(a - b)} = \frac{\cot b + \cot a}{\cot b - \cot a} = \frac{\tan a + \tan b}{\tan a - \tan b};\qquad (362)$$

$$\frac{\sin(a \pm b)}{\cos(a \mp b)} = \frac{\cot b \pm \cot a}{\pm 1 + \cot a \cot b} = \frac{\tan a \pm \tan b}{1 \pm \tan a \tan b};\qquad (363)$$

$$\frac{\cos(a+b)}{\cos(a-b)}=\frac{\cot b-\tan a}{\cot b+\tan a}=\frac{1-\tan a\ \tan b}{1+\tan a\ \tan b}; \tag{364}$$

$$\frac{1+\sin q}{1-\sin q}=\tan^2(45^\circ+\tfrac{1}{2}q); \tag{365}$$

$$\frac{1+\cos p}{1-\cos p}=\cot^2\tfrac{1}{2}p; \tag{366}$$

$$\frac{1+\sin q}{1+\cos p}=\frac{\sin^2(45^\circ+\frac{1}{2}q)}{\cos^2\frac{1}{2}p};\ \frac{1-\sin q}{1-\cos q}=\frac{\sin^2(45^\circ-\frac{1}{2}q)}{\sin^2\frac{1}{2}q}; \tag{367}$$

$$\tan a\pm\tan b=\frac{\sin(a\pm b)}{\cos a\cos b};\ \cot a\pm\cot b=\frac{\sin(b\pm a)}{\sin a\sin b}; \tag{368}$$

$$\tan a\pm\cot b=\frac{\pm\cos(a\mp b)}{\cos a\sin b};\ \cot a\pm\tan b=\frac{\cos(a\mp b)}{\sin a\cos b}. \tag{369}$$

PROPOSITION X.

All denominate equations are homogeneous. (370)

A *denominate* equation is one involving denominate quantities, such as length, surface, volume, weight, time, velocity, and the like, referred indeed to a unit of measure, but distinguished from abstract quantities or mere numbers.

By clearing of fractions and transposing, it is evident that any abstract or numerical equation may be represented by

$$abc\ldots[n\text{ factors}]+a_2b_2c_2\ldots[n_2\text{ factors}]+a_3b_3c_3\ldots[n_3]+\ldots=0;$$

for, if there were any powers they would be embraced in the products of equal factors, such as $ab=aa=a^2$, $abc=aaa=a^3\ldots$; $a_2\cdot b_2=a_2\cdot a_2=a_2^2,\ldots$, &c.; and we may suppose the equation freed from radicals by involution. Now $a, b, c, \ldots, a_2, b_2\ldots$, being *numbers*, may represent the quotients of any denominate quantities divided by their unit of measure, or we may have

$$a=\frac{A}{M},\ b=\frac{B}{M},\ c=\frac{C}{M},\ldots;\ a_2=\frac{A_2}{M},\ldots;$$

thus, if $A=$ a line 15 feet in length and the unit of measure be one yard, $\frac{A}{M}=\frac{15\text{ feet}}{3\text{ feet}}=$ the number 5; and so on. Substituting these values in the equation above, we have

$$\frac{A}{M}\cdot\frac{B}{M}\cdot\frac{C}{M}\cdot\ldots[n.]+\frac{A_2}{M}\cdot\frac{B_2}{M}\cdot\ldots[n_2]+\ldots=0,$$

or $$ABC\ldots[n]+A_2B_2C_2\ \ldots\ [n_2]\cdot M^{n-n_2}+A_3B_3\ \ldots\ [n_3]\cdot M^{n-n_3}+\ldots=0,$$

clearing of fractions, on the supposition that the equation has been arranged so that $n > n_2 > n_3 > \dots$. The last equation is homogeneous, being of the nth degree, or containing n factors in each term, and, as it is denominate, the proposition is demonstrated.

This theorem may be serviceable to those not yet well practised in algebra, by detecting errors. For, if we begin a problem with a denominate equation, all the following equations being denominate, will be homogeneous, and if any one, as that containing the result, want this homogeneity, it is an index of error in the operation.

But a more important application is the restoring of a quantity which has disappeared from a denominate equation by being assumed as the unit of measure. Thus, if $M = 1$, the above equation becomes

$$ABC \cdot \dots [n] + A_2B_2C_2 \cdot \dots [n_2] + A_3B_3C_3 \cdot \dots [n_3] + \dots = 0,$$

and the homogeneity disappears; to restore it, however, it is obviously necessary and sufficient to introduce the unit of measure, M, as a factor, into each term affected by an exponent which is the deficiency of the term in degree.

Suppose we are to restore the radius in (317); the first member, $\sin x$, is of the first degree, every term of the second must be the same, which requires

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3r^2} + \frac{x^5}{1 \dots 5r^4} - \frac{x^7}{1 \dots 7r^6} +, -, \dots .$$

The first equation in (320) becomes

$$r \sin(a + x) = \sin a \cos x + \cos a \sin x,$$

when the radius is restored. So (346) when radius $= r$, is

$$\tan(a + b) = \frac{r^2(\tan a + \tan b)}{r^2 - \tan a \tan b}.$$

It is recommended to the student, as an exercise, to restore the radius in all the preceding forms, and to inspect the geometrical equations which have occurred in regard to their homogeneity.

PROPOSITION XI.

It is required to develope the arc in terms of its tangent.

Let the arc AB, as it is the function, be indicated by y and its tangent AT, being the independent variable, by x; it is required

to find the function $y=fx$, that y is of x. Give to x the vanishing increment h, to y the corresponding increment k, and draw l through T perpendicular to TO and terminating in the secant drawn through the extremities of k and h. Similar triangles give us (300)

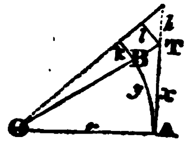


Fig. 56.

$$\left[\frac{k}{l}\right]=\frac{\text{OB}}{\text{OT}},\ \left[\frac{l}{h}\right]=\frac{\text{OA}}{\text{OT}};\ \therefore \left[\frac{k}{l}\right]\cdot\left[\frac{l}{h}\right]=\frac{\text{OA}\cdot\text{OB}}{\text{OT}^2}=\frac{r^2}{r^2+x^2},$$

$$\text{or}\quad y'=\left[\frac{k}{h}\right]=\frac{r^2}{r^2+x^2}=1-\frac{x^2}{r^2}+\frac{x^4}{r^4}-\frac{x^6}{r^6}+,-,\ldots;$$

whence, returning to the function (246) we have

$$y=x-\tfrac{1}{3}\cdot\frac{x^3}{r^2}+\tfrac{1}{5}\cdot\frac{x^5}{r^4}-\tfrac{1}{7}\cdot\frac{x^7}{r^6}+,-,\ldots+\textit{constant};$$

but x and y vanish together,

$$\therefore\qquad 0=0+\textit{constant},\ \therefore\ \textit{constant}=0;$$

$$\therefore\qquad y=x-\tfrac{1}{3}\cdot\frac{x^3}{r^2}+\tfrac{1}{5}\cdot\frac{x^5}{r^4}-\tfrac{1}{7}\cdot\frac{x^7}{r^6}+,-,\ldots, \tag{371}$$

which is the required relation. When radius = 1, (271) becomes

$$y=\tan y-\tfrac{1}{3}\tan^3 y+\tfrac{1}{5}\tan^5 y-\tfrac{1}{7}\tan^7 y+,-,\ldots. \tag{372}$$

PROPOSITION XII.

To compute the semicircumference π (pi), *when radius is made unity.*

For this purpose Machin puts $\tan y=\frac{1}{5}$;

$$\therefore (349)\qquad \tan 2y=\frac{2\cdot\frac{1}{5}}{1-(\frac{1}{5})^2}=\frac{5}{12},$$

$$\therefore\qquad \tan 4y=\tan 2(2y)=\frac{2\cdot\frac{5}{12}}{1-(\frac{5}{12})^2}=\frac{120}{119};$$

but (353), $\tan 45^\circ=r=1$,

$$\therefore\ \tan(4y-45^\circ)=\frac{\tan 4y-\tan 45^\circ}{1+\tan 4y\tan 45^\circ}=\frac{1}{239};$$

whence, by substitution in (372), we get

$$\textit{arc}\ y_{\tan y=\frac{1}{5}}=\tfrac{1}{5}-\tfrac{1}{3}(\tfrac{1}{5})^3+,-,\ldots,$$

and $\textit{arc}(4y-45^\circ)_{\tan(4y-45^\circ)=\frac{1}{239}}=\frac{1}{239}-\frac{1}{3}(\frac{1}{239})^3+,-,\ldots;$

$$\therefore\ \tfrac{1}{4}\pi=45^\circ=4y-(4y-45^\circ)=4[\tfrac{1}{5}-\tfrac{1}{3}(\tfrac{1}{5})^3+,-,\ldots]$$
$$-[\tfrac{1}{239}-\tfrac{1}{3}(\tfrac{1}{239})^3+,-,\ldots].$$

Operation.

		+ Terms.	
1°.	$\frac{1}{5} = {\cdot}2$	·200000000000000	1,
3°.	$(\frac{1}{5})^5 = (\frac{1}{5})^3(\frac{1}{5})^2 = {\cdot}008 \cdot {\cdot}04 = {\cdot}00032$	·000064000000000	5
5°.	·000000512	·000000056888889	9
7°.	·0000000008192	·000000000063016	13
9°.	·0000000000131072	·000000000000077	17
		·200064056951982	
		− Terms.	
2°.	$(\frac{1}{5})^3 = (\frac{1}{5})(\frac{1}{5})^2 = {\cdot}2 \cdot {\cdot}04 = {\cdot}008$	·002666666666667	3
4°.	·0000128	·000001828571429	7
6°.	·00000002048	·000000001861818	11
8°.	·000000000032768	·000000000002185	15
10°.	·0000000000000524288	·000000000000003	19
		·002668497102102	
	dif.	·197395559849880	
		4	
	$4[\frac{1}{5} - \frac{1}{3}(\frac{1}{5})^3 +, -, \ldots] =$ +	·789582239399520	
	$-[\frac{1}{239} - \frac{1}{3}(\frac{1}{239})^3 +, -, \ldots] = \begin{cases} - \\ + \\ - \end{cases}$	·004184100418410 ·000000024416592 ·000000000000256	
	$\frac{1}{4}\pi =$	·785398163397446	
	$\therefore\ \pi = 3$	·14159265358979[84]	
	or $\pi = 3$	·141592653589793,	(373)

The order of computation is indicated by 1°, 2°, 3°, ….

correcting the last digits by an extension of the work.

Cor. $\frac{1}{2}\ = 90° = 1{\cdot}570796326794896,$ (374)

$\frac{1}{6}\pi = 30° = 0{\cdot}523598775598299,$

$10° = 0{\cdot}174532925199433,$

$1° = 0{\cdot}017453292519943,$

$6' = 0°{\cdot}1 = 0{\cdot}001745329251994.$

PROPOSITION XIII.

To compute the trigonometrical lines.

Combining (374) with (317) and (318) we obtain

$$\sin 1° = ({\cdot}0174533) - \frac{({\cdot}0174533)^3}{1 \cdot 2 \cdot 3} +, -, \ldots = {\cdot}01745,$$

and $\cos 1° = 1 - \frac{(\cdot 0174533)^2}{1 \cdot 2} +, - \ldots = \cdot 99985.$

Hence, (315), (316),

$$\sin 2° = \sin(1° + 1°) = \cdot 01745\left[1 - \frac{(\cdot 01 \ldots)^2}{2} +, -, \ldots\right]$$
$$+ \cdot 99985\left[(\cdot 01 \ldots) - \frac{(\cdot 01 \ldots)^3}{1 \cdot 2 \cdot 3} +, -, \ldots\right] = \cdot 03490,$$
$$\cos 2° = \cdot 99985\left[1 - \frac{(\cdot 01 \ldots)^2}{1 \cdot 2} +, -, \ldots\right]$$
$$- \cdot 01745\left[(\cdot 01 \ldots) - \frac{(\cdot 01 \ldots)^3}{1 \cdot 2 \cdot 3} +, -, \ldots\right] = \cdot 99939;$$
$$\sin 3° = \sin(2° + 1°) = \cdot 05234,$$
$$\cos 3° = \cos(2° + 1°) = \cdot 99863;$$

&c. &c. &c.

The processes just indicated may be advantageously used, the first for the computation of the sine or cosine of a small arc when a large number of decimal places is required; the second for interpolating between sines already calculated and set down in a table; but, as two multiplications in each operation are requisite, a better method may be employed in making up a table, where a single constant multiplier will be sufficient.

In (329), making $p = (m + n)a$, $q = (m - n)a$, we have

$$\sin(m + n)a + \sin(m - n)a = 2 \sin ma \cos na, \qquad (375)$$

a form that will be occasionally serviceable. If, for instance, we make $m = 1$, $n = 1$, we get

$$\sin 2a = 2 \sin a \cos a,$$

a form already obtained; $m = 2$, $n = 1$, gives

$$\begin{aligned} \sin 3a &= 2 \sin 2a \cos a - \sin a \\ &= 4 \sin a \cos^2 a - \sin a \\ &= 4 \sin a(1 - \sin^2 a) - \sin a \\ &= 3 \sin a - 4 \sin^3 a. \end{aligned}$$

The student may put $m = 3$, $n = 1$; $m = 4$, $n = 1$; &c.; $m = 3$, $n = 2$; $m = 4$, $n = 2$; $m = 4$, $n = 3$; &c.; and find the results. If we make $a = 1$, we get

$$\sin(m + n) + \sin(m - n) = 2 \sin m \cos n,$$

which might have been obtained by adding in (320); making $n = 1°$ and putting $r =$ the constant multiplier $2 \cos 1°$, we have

$$\sin(m + 1) = r \sin m - \sin(m - 1),$$

a convenient form for computing the sines. Making $m = 1°, 2°, 3°, \ldots$, we have

$$\begin{aligned}\sin 2° &= r\sin 1° - 0,\\ \sin 3° &= r\sin 2° - \sin 1°,\\ \sin 4° &= r\sin 3° - \sin 2°,\\ \&c. &\quad \&c.\end{aligned}$$

It is recommended to the student to find r correct to 7 or 8 places, and execute the computation just indicated, employing the method of shortened multiplication.

We proceed to show in what way certain sines may be expressed in finite terms.

In (327) and (328), making $x = 45°$, we have

$$(1+1)^{\frac{1}{2}} + (1-1)^{\frac{1}{2}} = 2\cos 45°, \;\therefore\; \cos 45° = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

and $$(1+1)^{\frac{1}{2}} - (1-1)^{\frac{1}{2}} = 2\sin 45°, \;\therefore\; \sin 45° = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

In (331) making $p = (m+n)a$, $q = (m-n)a$, we have

$$\cos(m+n)a + \cos(m-n)a = 2\cos ma \cos na, \tag{376}$$

$\therefore$ $\cos(1+1)a + \cos(1-1)a = 2\cos a \cos a$,

or $\cos 2a = 2\cos^2 a - 1$;

$\therefore$ $\cos(2+1)a = 2\cos 2a \cos a - \cos(2-1)a$,

or $\cos 3a = 4\cos^3 a - 3\cos a$.

If, in the last form, we put $a = 30°$, there results

$$\cos 90° = 4\cos^3 30° - 3\cos 30°,$$

or $$0 = 4\cos^3 30° - 3\cos 30°.$$

$\therefore$ $$0 = 4\cos^2 30° - 3,$$

$\therefore$ $$\sin 60° = \cos 30° = \frac{\sqrt{3}}{2};$$

$\therefore$ $$\cos 60° = \sin 30° = (1 - \sin^2 60°)^{\frac{1}{2}} = \tfrac{1}{2}.$$

Again, $$\begin{aligned}4\cos^3 18° - 3\cos 18° &= \cos 3 \cdot 18° = \cos 54°\\ &= \sin 36° = 2\sin 18° \cos 18°,\end{aligned}$$

$\therefore$ $4\cos^2 18° - 3 = 2\sin 18°$,

$\therefore$ $4(1 - \sin^2 18°) - 3 = 2\sin 18°$,

or $4 - 4\sin^2 18° - 3 = 2\sin 18°$,

$\sin^2 18° + \tfrac{1}{2}\sin 18° = \tfrac{1}{4}$,

$\therefore$ $$\cos 72° = \sin 18° = \frac{\sqrt{5}-1}{4};$$

$\therefore$ $$\sin 72° = \cos 18° = (1 - \sin^2 18)^{\frac{1}{2}} = \frac{(10 + 2 \cdot 5^{\frac{1}{2}})^{\frac{1}{2}}}{4}.$$

In (328), making $x = 15^\circ$, and observing that the minus sign is to be employed when $x < 45^\circ$, we have

$$(1+\tfrac{1}{2})^{\frac{1}{2}} - (1-\tfrac{1}{2})^{\frac{1}{2}} = 2\sin 15^\circ,$$

$$\therefore \qquad \cos 75^\circ = \sin 15^\circ = \frac{1}{2\sqrt{2}}(\sqrt{3}-1).$$

The same form will give us the sine of 9°, then

$$\text{sine } 3^\circ = \sin(18^\circ - 15^\circ) = \sin 18^\circ \cos 15^\circ - \cos 18^\circ \sin 15^\circ,$$

and the cosine of 3° being known, the following table of sines and cosines may be calculated (320)*

$$\sin 3^\circ = \cos 87^\circ = \frac{3^{\frac{1}{2}}+1}{8 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}-1) - \frac{3^{\frac{1}{2}}-1}{8}(5+5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 6^\circ = \cos 84^\circ = -\frac{1}{8}(5^{\frac{1}{2}}-1) + \frac{3^{\frac{1}{2}}}{4 \cdot 2^{\frac{1}{2}}}(5-5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 9^\circ = \cos 81^\circ = \frac{1}{4 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}+1) - \frac{1}{4}(5-5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 12^\circ = \cos 78^\circ = -\frac{3^{\frac{1}{2}}}{8}(5^{\frac{1}{2}}-1) + \frac{1}{4 \cdot 2^{\frac{1}{2}}} \cdot 5^{\frac{1}{2}} + 5^{\frac{1}{2}}.$$

$$\sin 15^\circ = \cos 75^\circ = \frac{1}{2 \cdot 2^{\frac{1}{2}}}(3^{\frac{1}{2}}-1).$$

$$\sin 18^\circ = \cos 72^\circ = \frac{1}{4}(5^{\frac{1}{2}}-1).$$

$$\sin 21^\circ = \cos 69^\circ = -\frac{3^{\frac{1}{2}}-1}{8 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}+1) + \frac{3^{\frac{1}{2}}+1}{8} \cdot 5^{\frac{1}{2}} - 5^{\frac{1}{2}}.$$

$$\sin 24^\circ = \cos 66^\circ = \frac{3^{\frac{1}{2}}}{8}(5^{\frac{1}{2}}+1) - \frac{1}{4 \cdot 2^{\frac{1}{2}}} \cdot 5^{\frac{1}{2}} - 5^{\frac{1}{2}}.$$

$$\sin 27^\circ = \cos 63^\circ = -\frac{1}{4 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}-1) + \tfrac{1}{4}(5+5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 30^\circ = \cos 60^\circ = \tfrac{1}{2}.$$

$$\sin 33^\circ = \cos 57^\circ = \frac{3^{\frac{1}{2}}+1}{8 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}-1) + \frac{3^{\frac{1}{2}}-1}{8}(5+5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 36^\circ = \cos 54^\circ = \frac{1}{2 \cdot 2^{\frac{1}{2}}}(5-5^{\frac{1}{2}})^{\frac{1}{2}}.$$

$$\sin 39^\circ = \cos 51^\circ = \frac{3^{\frac{1}{2}}+1}{8 \cdot 2^{\frac{1}{2}}}(5^{\frac{1}{2}}+1) - \frac{3^{\frac{1}{2}}-1}{8}(5-5^{\frac{1}{2}})^{\frac{1}{2}}.$$

* Library of Useful Knowledge.

$$\sin 42^\circ = \cos 48^\circ = -\tfrac{1}{8}(5^{\frac12}-1) + \frac{3^{\frac12}}{4 \cdot 2^{\frac12}}(5+5^{\frac12})^{2}.$$

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{2^{\frac12}}.$$

$$\sin 48^\circ = \cos 42^\circ = \frac{-3^{\frac12}}{8}(5^{\frac12}-1) + \frac{1}{4 \cdot 2^{\frac12}}(5+5^{\frac12})^{\frac12}.$$

$$\sin 51^\circ = \cos 39^\circ = \frac{3^{\frac12}-1}{8 \cdot 2^{\frac12}}(5^{\frac12}+1) + \frac{3^{\frac12}+1}{8}(5-5^{\frac12})^{\frac12}.$$

$$\sin 54^\circ = \cos 36^\circ = \tfrac{1}{4}(5^{\frac12}+1).$$

$$\sin 57^\circ = \cos 33^\circ = -\frac{3^{\frac12}-1}{8 \cdot 2^{\frac12}}(5^{\frac12}-1) + \frac{3^{\frac12}+1}{8}(5+5^{\frac12})^{\frac12}.$$

$$\sin 60^\circ = \cos 30^\circ = \frac{3^{\frac12}}{2}.$$

$$\sin 63^\circ = \cos 27^\circ = \frac{1}{4 \cdot 2^{\frac12}}(5^{\frac12}-1) + \tfrac{1}{4}(5+5^{\frac12})^{\frac12}.$$

$$\sin 66^\circ = \cos 24^\circ = \tfrac{1}{8}(5^{\frac12}+1) + \frac{3^{\frac12}}{4 \cdot 2^{\frac12}}(5-5^{\frac12})^{\frac12}.$$

$$\sin 69^\circ = \cos 21^\circ = \frac{3^{\frac12}+1}{8 \cdot 2^{\frac12}}(5^{\frac12}+1) + \frac{3-1^{\frac12}}{8}(5-5^{\frac12})^{\frac12}.$$

$$\sin 72^\circ = \cos 18^\circ = \frac{1}{2 \cdot 2^{\frac12}}(5+5^{\frac12})^{\frac12}.$$

$$\sin 75^\circ = \cos 15^\circ = \frac{1}{2 \cdot 2^{\frac12}}(3^{\frac12}+1).$$

$$\sin 78^\circ = \cos 12^\circ = \tfrac{1}{8}(5^{\frac12}-1) + \frac{3^{\frac12}}{4 \cdot 2^{\frac12}}(5+5^{\frac12})^{\frac12}.$$

$$\sin 81^\circ = \cos 9^\circ = \frac{1}{4 \cdot 2^{\frac12}}(5^{\frac12}+1) + \tfrac{1}{4}(5-5^{\frac12})^{\frac12}.$$

$$\sin 84^\circ = \cos 6^\circ = \frac{3^{\frac12}}{8}(5^{\frac12}+1) + \frac{1}{4 \cdot 2^{\frac12}}(5-5^{\frac12})^{\frac12}.$$

$$\sin 87^\circ = \cos 3^\circ = \frac{3^{\frac12}-1}{8 \cdot 2^{\frac12}}(5^{\frac12}-1) + \frac{3^{\frac12}+1}{8}(5+5^{\frac12})^{\frac12}.$$

$$\sin 90^\circ = \cos 0^\circ = 1.$$

Other sines may be interpolated by (315) or by the method of differences, and the tangents will be found by dividing the sine by the cosine.

It is recommended to the student to execute several of the computations indicated above, carrying out the work to ten or fifteen decimal places, and employing a shortened method of extracting

the square root as well as in multiplying and dividing. It will be advantageous to free the denominators of radicals; for example, let it be required to find the sine of 75°.

Operation.

$$\sin 75^\circ = \frac{1}{2 \cdot 2^{\frac{1}{2}}}(3^{\frac{1}{2}}+1) = \frac{2^{\frac{1}{2}}(3^{\frac{1}{2}}+1)}{2 \cdot 2}.$$

```
      3)1·7320508076            2^½ = 1·4142135624
        1                             2·7320508076
       ———                            ————————————
    27)200                            2·8284271248
        189                            9899495937
       ———                              424264069
   343)1100                              28284271
        1029                               707107
        ———                                 11313
   3462)7100                                   99
        6924                                    8
        ———                           ————————————
3|4|6|4|0|5)1760000                 4)3·8637034045
            1732025                   ————————————
            ———————                   0·9659258511 = sin75°.
              27975
              27712
              —————
                263
                242
                ———
                 21
                 21
                 ——
```

How many digits may be depended on?

But a table of sines is comparatively of little importance, as the logarithms of these numbers are generally preferable in practice.

PROPOSITION XIV.

To compute the logarithmic sines and tangents.

Restricting (317) to the fourth power of x, we have

$$\sin x = x(1 - \tfrac{1}{6}x^2 + \tfrac{1}{120}x^4),$$

$$\therefore \log. \sin x = \log. x + \log. (1 - \tfrac{1}{6}x^2 + \tfrac{1}{120}x^4)$$

$$= \log. x + M[(-\tfrac{1}{6}x^2 + \tfrac{1}{120}x^4)^1 - \tfrac{1}{2}(-\tfrac{1}{6}x^2)^2] \quad [(270)]$$

$$= \log. x - \frac{M}{6} \cdot x^2\left(1 + \frac{x^2}{30}\right),$$

$$\therefore \quad \log.\sin x = \log. x - \text{No. to } \{\bar{2}{\cdot}8596331 + 2\log. x + \log.[1 + \text{No. to } (2\log. x - 1{\cdot}477)]\}. \qquad (378)$$

As an example, let it be required to find the logarithm of the sine of 5°.

We have (374)

$$x = 5^\circ = {\cdot}08726646\,;$$

$$\therefore \quad \log. x = \bar{2}{\cdot}9408474$$

$$\begin{array}{rl|l}
2\log. x = & \bar{3}{\cdot}8816948 & \bar{3}{\cdot}882 \\
& \bar{2}{\cdot}8596331 & 1{\cdot}477 \\
\log. 1{\cdot}000254 = & 0{\cdot}0000103 & \bar{4}{\cdot}405 \\
& \bar{4}{\cdot}7413382 & \text{No. } {\cdot}000254 \\
\text{No.} & 0{\cdot}0005512 &
\end{array}$$

$$\therefore \quad \log.\sin 5^\circ = \bar{2}{\cdot}9402962$$

Form (378) should not be employed when the arc exceeds 5°, and the last term, log. $[1 + \text{No. to } (2\log. x - 1{\cdot}447)]$, may be omitted if the arc be less than 3°.

Imitating the process above, we find

$$\log.\cos x = -\text{No. to } \{\bar{1}{\cdot}3367543 + 2\log. x + \log.[1 + \text{No. to } (2\log. x - 0{\cdot}778)]\}. \qquad (379)$$

Operating as in the last example, we obtain

$$\log.\cos 5^\circ = -0{\cdot}0016558 = \bar{1}{\cdot}9983442.$$

The logarithmic tangent will be found from the relation

$$\tan = \frac{\text{sine}}{\text{cosine}}, \text{ or } \log.\tan = \log.\sin - \log.\cos,$$

and the logarithmic cotangent results from

$$\cot = \frac{\cos}{\sin}, \text{ or } \log.\cot = \log.\cos - \log.\sin.$$

Thus $\quad \log.\sin 5^\circ = \bar{2}{\cdot}9402962,$

$\log.\cos 5^\circ = \bar{1}{\cdot}9983442\,;$

$\therefore \quad \log.\tan 5^\circ = \bar{2}{\cdot}9419520,$

and $\quad \log.\cot 5^\circ = 1{\cdot}0580480.$

In order to avoid minus characteristics, 10 is usually added; thus in most tables we find log. sin5° = 8·9402960.

Dividing the first of (320) by sin$a$, there results

$$\frac{\sin(a+x)}{\sin a} = \cos x + \cos a \sin x = \cos x(1 + \cot a \tan x),$$

$\therefore$ $\log.\ \sin(a+x) = \log.\ \sin a + \log.\ \cos x + M(\cot a \tan x$
$- \tfrac{1}{2}\cot^2 a \tan^2 x +, -, \ldots).$ (380)

By a similar process we find

$\log.\ \cos(a+x) = \log.\ \cos a + \log.\ \cos x - M(\tan a \tan x$
$+ \tfrac{1}{2}\tan^2 a \tan^2 x +, -, \ldots).$ (381)

Let $a = 5^\circ$ and $x = 0^\circ{\cdot}1 = 6'$, then

$+\log.\ \sin a = \bar{2}{\cdot}9402962$	$\log.\ M$	$= \bar{1}{\cdot}6377843$	(1)
$+\log.\ \cos x = \bar{1}{\cdot}9999993$	$\log.\ \cot a$	$= 1{\cdot}0580482$	(2)
$+\ \text{No.} = 0{\cdot}0086639$	$\log.\ \tan x$	$= \bar{3}{\cdot}2418778$	(3)
$-\tfrac{1}{2}\text{No.} = 0{\cdot}0000864$		$\bar{3}{\cdot}9377103$	
$+\tfrac{1}{3}\text{No.} = 0{\cdot}0000011$		$\bar{4}{\cdot}2376363$	$2[(1)+(2)]+(1)$
$\therefore \log.\sin 5^\circ{\cdot}1 = \bar{2}{\cdot}9488741$		$\bar{6}{\cdot}5375623$	$3[(1)+(2)]+(1)$

Calculate the log. cos$5^\circ{\cdot}1$.

In order to avoid the accumulation of errors, the computations should be recommenced from new points of departure, for which purpose the above table of sines and cosines to every 3° may be employed.

PROPOSITION XV.

To develope the arc in terms of its sine.

We have

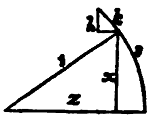


Fig. 57.

$$y' = \left[\frac{k}{h}\right] = \frac{1}{z} = \frac{1}{(1-x^2)^{\frac{1}{2}}} = (1-x^2)^{-\frac{1}{2}},$$

or (250), $y' = (1)^{-\frac{1}{2}} + (-\tfrac{1}{2})\,(1)^{-\frac{1}{2}-1} \cdot (-x^2)^1$

$$+ (-\tfrac{1}{2})\,(-\tfrac{1}{2}-1)\,(1)^{-\frac{1}{2}-2} \cdot \frac{(-x^2)^2}{2} + \ldots,$$

or $$y' = 1 + \tfrac{1}{2}x^2 + \tfrac{1}{2} \cdot \frac{1+1\cdot 2}{2} \cdot \frac{1}{1\cdot 2} \cdot x^4$$

$$+ \tfrac{1}{2} \cdot \frac{1+1\cdot 2}{2} \cdot \frac{1+2\cdot 2}{2} \cdot \frac{1}{1\cdot 2\cdot 3} \cdot x^6$$

$$+ \tfrac{1}{2} \cdot \frac{1+1\cdot 2}{2} \cdot \frac{1+2\cdot 2}{2} \cdot \frac{1+3\cdot 2}{2}$$

$$\cdot \frac{1}{1\cdot 2\cdot 3\cdot 4} \cdot x^8 + \ldots;$$

$$\therefore \quad y = x + \tfrac{1}{3}\cdot\tfrac{1}{2}\cdot x^3 + \tfrac{1}{5}\cdot\tfrac{1}{2}\cdot\frac{1+1\cdot2}{2}\cdot\frac{1}{1\cdot2}\cdot x^5$$
$$+\tfrac{1}{7}\cdot\tfrac{1}{2}\cdot\frac{1+1\cdot2}{2}\cdot\frac{1+2\cdot2}{2}\cdot\frac{1}{1\cdot2\cdot3}\cdot x^7$$
$$+\tfrac{1}{9}\cdot\tfrac{1}{2}\cdot\frac{1+1\cdot2}{2}\cdot\frac{1+2\cdot2}{2}\cdot\frac{1+3\cdot2}{2}$$
$$\cdot\frac{1}{1\cdot2\cdot3\cdot4}\cdot x^9 + \ldots; \qquad (382)$$

where no constant is to be added, since x and y vanish together.

PROPOSITION XVI.

To resolve the equation.

$$a\sin x + b\cos x = c.$$

Putting $$\tan z = \frac{\sin z}{\cos z} = \frac{b}{a}, \qquad (383)$$

we have $$\sin x + \frac{\sin z}{\cos z}\cdot\cos x = \frac{c}{a},$$

$$\therefore \quad \sin x\cos z + \cos x\sin z = \frac{c\cos z}{a},$$

or $$\sin(x+z) = \frac{c\cos z}{a}. \qquad (384)$$

(383) makes known z, then $(x+z)$ is determined by (384) and finally x. For an example, let

$$\sqrt{216}\cdot\sin x + \sqrt{72}\cdot\cos x = 12; \text{ we find } x = 15°.$$

PROPOSITION XVII.

To resolve the equation.

$$\sin(x+k) = m\sin(x+l).$$

We have (362),

$$m = \frac{\sin(x+k)}{\sin(x+l)} = \frac{\sin[x+\frac{1}{2}(k+l)+\frac{1}{2}(k-l)]}{\sin[x+\frac{1}{2}(k+l)-\frac{1}{2}(k-l)]}$$
$$= \frac{\tan[x+\frac{1}{2}(k+l)]+\tan\frac{1}{2}(k-l)}{\tan[x+\frac{1}{2}(k+l)]-\tan\frac{1}{2}(k-l)};$$

$$\left.\begin{array}{ll} \therefore & \tan[x+\frac{1}{2}(k+l)] = \dfrac{1+m}{1-m}\tan\frac{1}{2}(l-k), \\ \text{or} & \tan[x+\frac{1}{2}(k+l)] = \tan(45°+v)\tan\frac{1}{2}(l-k), \\ \text{putting (354)} & m = \tan v. \end{array}\right\} (385)$$

The second form will be preferable, when m is such a quantity as to be most readily computed by logarithms. What will the equations become when $k=0$? when $l=0$? The student may form an example for himself

PROPOSITION XVIII.

To resolve the quadratic equation,

$$x^2+2px=q,$$

when p *and* q *are such as to require logarithmic tables.*

We have

$$x+p=\pm(p^2+q)^{\frac{1}{2}}=\pm(p^2+p^2\tan^2 v)^{\frac{1}{2}} \quad [\text{putting } q=p^2\tan^2 v]$$
$$=\pm p(1+\tan^2 v)^{\frac{1}{2}}=\pm p\sec v,$$

or $$x+p=\pm\frac{p}{\cos v},\ \tan v=\frac{q^{\frac{1}{2}}}{p}. \qquad (385_2)$$

Example. Given

$$x^2+2\cdot\frac{25{\cdot}35\sin 86{\cdot}25^\circ}{\sin 161{\cdot}75^\circ}\cdot x=\frac{2\cdot 357\sin 86{\cdot}25^\circ}{\sin 75{\cdot}5^\circ\sin 161{\cdot}75^\circ},$$

to find x.

SECTION SECOND.

Resolution of Triangles and Mensuration of Heights and Distances.

PROPOSITION I.

The length of a line and its inclination to a second line being given, to find its PROJECTION *upon that line.*

Let a be the given line and l the line upon which its projection is to be made; drop the perpendiculars p, p_2 from the extremities of a upon l; then a_2, the portion of l intercepted

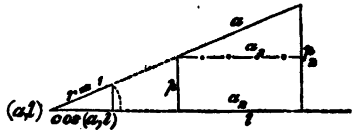


Fig. 58.

between these perpendiculars, is called the *projection of* a *upon* l. Through the extremity of a nearest l, draw a_3 parallel to a_2 and terminating in p_2, then $a_2=a_3$. Also produce a to intersect l,

making the angle (a,l); from the angular point and on the production of a, measure off the radius, $r = 1$, and from the extremity of r drop the perpendicular intercepting the cosine, $\cos(a,l)$; then

$$a_2 : a :: \cos(a,l) : r,$$

or

$$a_2 : a :: \cos(a,l) : 1;$$

$$\therefore \quad a_2 = a\cos(a,l), \text{ i. e.,}$$

The projection of a line is found by multiplying the line into the cosine of its inclination to the line upon which it is projected. (386)

PROPOSITION II.

To find an equation as simple as possible that shall embrace the relation existing between the sides and angles of a triangle.

Let the sides of any triangle be denoted by a, b, c, and the angles respectively opposite by A, B, C. Then, dropping a perpendicular from C, we have (386)

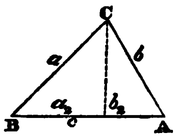


Fig. 59.

$$a_2 = a\cos B,$$

and

$$b_2 = b\cos A;$$

$$\therefore \quad c = a_2 + b_2 = a\cos B + b\cos A, \text{ i. e.,}$$

Either side of a triangle is equal to the sum of the products formed by multiplying the two remaining sides into the cosines of their respective inclinations to the first mentioned line. (387)

This proposition, obviously little else than a corollary from (386), may be regarded as the fundamental theorem in the resolution of triangles; since it gives at once the equations,

$$\left.\begin{aligned} a &= b\cos C + c\cos B, \\ b &= a\cos C + c\cos A, \\ c &= a\cos B + b\cos A; \end{aligned}\right\} \qquad (387_2)$$

from which, by elimination, all possible relations among the sides and angles may be drawn.

If one of the angles, as A, become greater than 90°, the corresponding side, $b_2 = b\cos A$, will be minus (338) $\therefore c = a_2 - b_2 = a\cos B - b\cos A$, and the theorem still holds good.

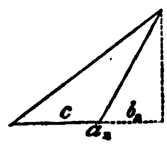


Fig. 59_2.

In order to find an equation embracing but the single angle A and the sides a, b, c, eliminating $\cos C$ between the first and second of (387_2), and multiplying the third by c, we have

$$a^2-b^2=ac\cos B-bc\cos A,$$

and $$c^2=ac\cos B+bc\cos A;$$

$\therefore$ $$c^2-a^2+b^2=2bc\cos A,$$

or $$b^2+c^2=a^2+2bc\cos A;$$ i. e.

PROPOSITION III.

The sum of the squares of any two sides of a triangle is equal to the squares of the third side increased by the double product of those two sides multiplied into the cosine of the angle which they include. (388)

$$\left.\begin{aligned} a^2+b^2&=c^2+2ab\cos C\\ a^2+c^2&=b^2+2ac\cos B\\ b^2+c^2&=a^2+2bc\cos A \end{aligned}\right\} \qquad (388_2)$$

*Cor.** $a^2+b^2+c^2=2ab\cos C+2ac\cos B+2bc\cos A.$ (389)

PROPOSITION IV.

To transform (388) *so as to be convenient for the logarithmic computation of* A.

Combining (388) and (324) we have

$$\frac{b^2+c^2-a^2}{2bc}=\cos A=1-2\sin^2\tfrac{1}{2}A,$$

$$\therefore \quad 2\sin^2\tfrac{1}{2}A=1+\frac{a^2-(b^2+c^2)}{2bc}=\frac{a^2-(b-c)^2}{2bc}$$

$$=\frac{(a+b-c)\ (a-b+c)}{2bc};$$

$$\therefore \quad \sin\tfrac{1}{2}A=\left[\frac{(a+b-c)\ (a+c-b)}{2b\ .\ 2c}\right]^{\frac{1}{2}}$$

$$=\left[\frac{(h-b)\ (h-c)}{bc}\right]^{\frac{1}{2}}, \qquad (390)$$

putting $h=\frac{1}{2}(a+b+c)$.

So $$\cos\tfrac{1}{2}A=\left[\frac{h(h-a)}{bc}\right]^{\frac{1}{2}}, \qquad (391)$$

---

* Express in words.

and ∴ $\tan. \frac{1}{2}A = \left[\frac{(h-b)(h-c)}{h(h-a)}\right]^{\frac{1}{2}}$. (392)

Also, taking the double product of (390) and (391), we have (321),

$$\sin A = \frac{2[h(h-a)(h-b)(h-c)]^{\frac{1}{2}}}{bc}; \qquad (393)$$

∴ $$\frac{\sin A}{a} = \frac{2[h(h-a)(h-b)(h-c)]^{\frac{1}{2}}}{abc},$$

so $$\frac{\sin B}{b} = \frac{2[h(h-b)(h-a)(h-c)]^{\frac{1}{2}}}{bac};$$

∴ $$\frac{\sin A}{a} = \frac{\sin B}{b}, \text{ or } a : b :: \sin A : \sin B; \text{ i. e.}$$

PROPOSITION V.

The sides of a triangle are to each other as the sines of the opposite angles. (394)

If the angle $C = 90°$, or B be the complement of A, then

$$\frac{a}{b} = \frac{\sin A}{\sin B} = \frac{\sin A}{\cos A} = \tan A, \text{ or}$$

Cor. $a = b \tan A$. (395)

Again (394) gives (40,3°)

$$a + b : a - b :: \sin A + \sin B : \sin A - \sin B,$$

or (333) $a + b : a - b :: \tan\frac{1}{2}(A + B) : \tan\frac{1}{2}(A - B)$; i. e.

PROPOSITION VI.

The sum of any two sides of a triangle is to their difference, as the tangent of the half sum of the angles opposite to the tangent of half their difference. (396)

The above theorems are adequate to the solution of all problems in *Plane Trigonometry*; and these problems may be reduced to one or other of the four following cases:

CASE. I. The angles and one side of a triangle being given, to find the remaining parts.

Rule. As the sine of the angle opposite the given side,
Is to the sine of the angle opposite the required side;
So is the given side
To the required side.

1°. As an example under this case, let it be required to find the distance, x, of an object rendered inaccessible by the intervention of a river. For this purpose I measure a base line of 10 chains, and taking the angles at its extremities, I find them to be, the one 80°, the other 70°.

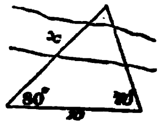


Fig. 60.

We have the angle at the object $= 180° - (80° + 70°) = 30°$;

$$\therefore \frac{\sin 30°}{\sin 70°} = \frac{10}{x}, \qquad \begin{array}{rrl} \log.\ \sin 30° = & \bar{1}{\cdot}69897 & [-] \\ \log.\ \sin 70° = & \bar{1}{\cdot}97299 & [+] \\ \log.\ 10 = & 1{\cdot}00000 & [+] \\ \hline \therefore \log.\ x = & 1{\cdot}27402 & \end{array}$$

$$\therefore x = 18{\cdot}794 \text{ chs.}$$

Case II. Two sides and an angle opposite one of them being given to find the remaining parts.

Rule. As the side opposite the given angle,
Is to the side opposite the required angle;
So is the sine of the given angle
To the sine of the required angle.

2°. To illustrate Case II., in the triangle ABC, let $AB = 13{\cdot}56$ chs., $BC = 7$ chs., and the angle $A = 25{\cdot}3°$; required $\angle$ C.

Operation.

$$\frac{7}{13{\cdot}56} = \frac{\sin 25{\cdot}3°}{\sin C}. \qquad \begin{array}{r} 0{\cdot}84510 \\ 1{\cdot}13226 \\ \bar{1}{\cdot}63079 \\ \hline \bar{1}{\cdot}91795 \end{array}$$

$$\therefore\ C = 55{\cdot}877°, \text{ or } = 180° - 55{\cdot}877 = 124{\cdot}123°.$$

The ambiguity of the angle C will be illustrated by dropping the perpendicular BP upon AC, and calculating its length; we have

$$\frac{\sin 90°}{\sin 25{\cdot}3°} = \frac{13{\cdot}56}{PB}; \qquad \begin{array}{r} 1{\cdot}00000 \\ \bar{1}{\cdot}63079 \\ 1{\cdot}13226 \\ \hline 0{\cdot}76305 \end{array}$$

$$\therefore\ BP = 5{\cdot}795;$$

whence, since BP is less than BC, taking the point C′ in AC and on the side of P opposite to C, so that PC′ shall = PC, and join-

ing BC′, we have BC′ = BC. Therefore the two triangles, ABC, ABC′, are alike compatible with the conditions of the problem; and the angles BC′A, BCA [= BC′C], are supplementary. If we had taken A = 44‘5°, we should have found log. sinC = 0‘13282, and, as a consequence, the sine of C greater than radius, which is impossible. The cause of impossibility will be manifest by computing the perpendicular BP, which, being found = 9‘5043, shows that BC is too short, when A = 44‘5°, to form a triangle. The student should make all the computations here indicated.

3°. In order to ascertain the altitude, LT, of a tower which I am prevented from approaching, I take a station, A, in the same horizontal plane with its foot, L, and observe the angle of elevation, TAL, at the top of the tower = 35°; then measuring AB = 113 feet directly back from A and in a line with L, I find the angle of elevation TBA = 25°

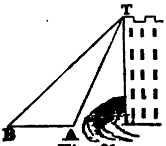


Fig. 61.

What is the altitude of the tower, and what was my distance from its foot when at the first station. *Ans.* LT = 157‘74.

4°. An individual, in order to fix the position of a certain point, P, in a harbor, selects a convenient place on shore and measures a base line, AB = 1970‘9 rods, and finds the angles, PAB = 100‘13°, PBA = 37°. Required AP, BP.

The student will compute AP, BP, and then verify by taking AP, or BP, for the known side and calculating AB.

5°. In order to determine the distance between two places, A and B, situated on opposite sides of a hill, and their relative altitudes, I measure the horizontal line AC = 15 rods, and take the angles of elevation PAK = 37‘9°, PCK = 30°, P being a flag on the summit. I then measure the base, BD = 11 rods, and take the angles of elevation PBL = 40°, PDM = 31‘1½°, also the angle of depression BDM = 25°.

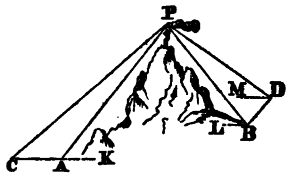


Fig. 62.

CASE III. Two sides and the included angle given, to determine the remaining parts.

Rule. As the sum of the two given sides
Is to their difference;
So is the tangent of the half sum of the opposite angles
To the tangent of their half difference.

6°. A surveyor, wishing to determine the side AB of a field, rendered incapable of direct measurement by reason of an intervening morass, runs the line AC, south 38° west, = 7·75 chains, then CB, south 25·8° east, 10·15 chains.

Produce AC in K and draw the meridian CS; then SCK = 38° and BCS = 25·8°,

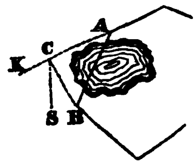


Fig. 63.

$$\therefore \quad \tfrac{1}{2}(\text{CAB}+\text{CBA})=\tfrac{1}{2}\text{BCK}=31{\cdot}9^\circ.$$

$$\therefore \quad \frac{10{\cdot}15+7{\cdot}75}{10{\cdot}15-7{\cdot}75} \qquad \begin{array}{l} 1{\cdot}25285 \\ 0{\cdot}38021 \end{array}$$

$$=\frac{\tan 31{\cdot}90}{\tan\frac{1}{2}(\text{A}-\text{B})}. \qquad \begin{array}{l} \bar{1}{\cdot}79410 \\ \hline \bar{2}{\cdot}92146 \end{array}$$

$$\therefore \quad \tfrac{1}{2}(\text{A}-\text{B})=\ \ 4^\circ\ 46'\ 46''$$

$$\text{and} \quad \tfrac{1}{2}(\text{A}+\text{B})=31^\circ\ 54'\ 00'';$$

$$\therefore \quad \text{A}=36^\circ\ 40'\ 46''.$$

The angle A being determined the solution will be readily finished, and we shall find

AB, S 1° 19′ 14″ W, 15·246 chs.

7°. Given the following courses,

1°. AB, N 46° E, 35 chs.
2°. BC, N 20° W, 55 chs.
3°. CD, S 35° W, 45 chs.,

to determine DA.

[Verify by employing the computed value of DA to find AB already known.]

Case IV. The sides given, to determine the angles.

Rule 1. $\log.\sin\frac{1}{2}\text{A}=\frac{1}{2}[\log.(h-b)+\log.(h-c)-(\log.b+\log.c)]$.

2. $\log.\cos\frac{1}{2}\text{A}=\frac{1}{2}[\log.h+\log.(h-a)-(\log.b+\log.c)]$.

3. $\log.\tan\frac{1}{2}\text{A}=\frac{1}{2}[\log.(h-b)+\log.(h-c)-\log.h-\log.(h-a)]$.

4. $\log.\sin\text{A}=\frac{1}{2}[\log.h+\log.(h-a)+\log.(h-b)+\log.(h-c)]$
$+\log.2-(\log.b+\log.c)$.

8°. Let it be required to find the angles of a triangular field, the sides of which are $300\frac{1}{4}$, $267\frac{3}{4}$, 199 feet.

First Operation.

$a = 300{\cdot}25$	$h - b = 115{\cdot}75$	$2{\cdot}06352$	[+]
$b = 267{\cdot}75$	$h - c = 184{\cdot}50$	$2{\cdot}26600$	[+]
$c = 199{\cdot}00$			
	$b = 267{\cdot}75$	$2{,}42773$	[−1]
$2)767{\cdot}00$	$c = 199{\cdot}00$	$2{\cdot}29885$	[−1]
$383{\cdot}50$		$2)\bar{1}{\cdot}60294$	

$$\therefore \quad \log.\ \sin\tfrac{1}{2}A = \bar{1}{\cdot}80147$$
$$\therefore \quad \tfrac{1}{2}A = 39^\circ\ 16'\ 40{\cdot}5''$$
$$\therefore \quad A = 78^\circ\ 33'\ 21''.$$

Second Operation.

$h = 383{\cdot}50$	$2{\cdot}58377$
$h - b = 115{\cdot}75$	$2{\cdot}06352$
$a = 300{\cdot}25$	$2{,}47748$
$c = 199{\cdot}00$	$2{\cdot}29885$
	$2)\bar{1}{\cdot}87096$

$$\therefore \quad \log.\ \cos\tfrac{1}{2}B = \bar{1}{\cdot}93548$$
$$\therefore \quad B = 60^\circ\ 55'\ 40''.$$

Third Operation.

$h - a = 83{\cdot}25$	$1{\cdot}92038$
$h - b = 115{\cdot}75$	$2{\cdot}06352$
$h = 383{\cdot}50$	$2{\cdot}58377$
$h - c = 184{\cdot}50$	$2{\cdot}26600$
	$2)\bar{1}{\cdot}13413$

$$\therefore \quad \log.\ \tan\tfrac{1}{2}C = \bar{1}{\cdot}56706$$
$$\therefore \quad C = 40^\circ\ 30'\ 42''$$

But $B = 60^\circ\ 55'\ 40''$

and $A = 78^\circ\ 33'\ 21''$;

$\therefore \quad A + B + C = 179^\circ\ 59'\ 43''$

diff. from $180^\circ\ 00'\ 00''$

by $0^\circ\ 00'\ 17''$ = the sum of errors.

The third operation for the computation of C is obviously un-

necessary, unless we wish to test the accuracy of the work. We have employed three methods in order to illustrate the rules; sometimes one and sometimes another will be preferable, according to the numbers.

9°. Required to determine the angles of a quadrilateral field from the following data:

$$AB = 56,\ BC = 76,\ CD = 87,\ DA = 43,\ BD = 67.$$

10°. Given two sides of a triangle 367·23, 273 chains, and the difference of the opposite angles 15·7°, to determine the triangle.

11°. Given the sum of two arcs and the ratio of their sines, to determine the arcs.

Let the arcs be denoted by u, v, their sum by e, and their ratio by r; we have

$$u + v = e,$$

and $$\sin v = r \sin u.$$

$$\therefore \quad \frac{\sin v}{\sin u} = r, \text{ which put} = \tan w = \frac{\tan w}{1};$$

$$\therefore \quad \frac{\sin u + \sin v}{\sin u - \sin v} = \frac{1 + r}{1 - r} = \frac{1 + \tan w}{1 - \tan w};$$

$$\therefore \quad \frac{\tan\frac{1}{2}(u + v)}{\tan\frac{1}{2}(u - v)} = \frac{1 + r}{1 - r} = \tan(45^\circ + w);$$

$$\therefore \quad \tan\tfrac{1}{2}(u - v) = \frac{1 - r}{1 + r} \cdot \tan\tfrac{1}{2}e = \frac{\tan\frac{1}{2}e}{\tan(45^\circ + w)},$$

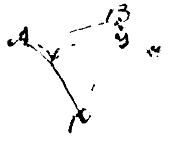
which makes known $\frac{1}{2}(u - v)$,
but $\frac{1}{2}(u + v) = \frac{1}{2}e$,
whence u and v are finally determined.

The student may make an application of the above in the solution of the following problem, taken from Davies' Legendre:

12°. "From a station, P, there can be seen three objects, A, B, and C, whose distances from each other are known, viz., AB = 800, AC = 600, and BC = 400 yards. There are also measured the horizontal angles, APC = 33° 45′, BPC = 22° 30′. It is required from these data to determine the three distances PA, PC, and PB."

The angles CAP and CBP will be the u and v of the eleventh. The student will make the computations, and devise means to satisfy himself of the correctness of his results.

13°. Wishing to ascertain the distance, AP, to an inaccessible object, P, also invisible from A, I measure to the right and left the



equal lines AB = AC = 21·37 chains, and the angles, BAC = 113° 12′, ABP = 65° 36′, ACP = 89° 5′.

How might the principle of this problem be applied to determine the distance of the moon, her zenith distances being observed by two astronomers, one at St. Petersburg, and the other at the Cape of Good Hope?

SECTION THIRD.

Quadrature of the Circle, the Ellipse, and Parabola.

PROPOSITION I.

To find the area of a circular sector in terms of its radius and arc.

The sector y is obviously a function of its arc x, the radius, r, being a constant quantity. Giving to x and y the vanishing increments, h and k, we have (311),

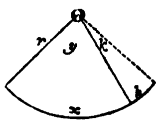


Fig. 64.

$$[k] = \tfrac{1}{2}r \cdot [h];$$

$\therefore$ $$y' = \left[\frac{k}{h}\right] = \tfrac{1}{2}r = \tfrac{1}{2}rx^0;$$

$\therefore$ (246), $$y = \tfrac{1}{2}rx + constant,$$

but $$y_{x=0} = 0;$$

$\therefore$ $$y = \tfrac{1}{2}rx, \text{ i. e.,}$$

The circular sector is measured by half the product of its radius and arc. (397)

Cor. The area of a circle is equal to the product of the radius multiplied into its semicircumference. (398)

Scholium. The celebrated *Problem of the Quadrature of the Circle** is evidently reduced to the following proposition:

PROPOSITION II.

The diameter of a circle being given, it is required to find the circumference.

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* See Montucla, "Histoire des Recherches sur la Quadrature du Cercle."

If in (371) we make $x = r$, there results

$$y = \tfrac{1}{4}circumference = (1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} +, -, \ldots)r;$$

$$\therefore \qquad (circumference)_r = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} +, -, \ldots) \cdot 2r.$$

So $\qquad (circumference)_{r_2} = 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} +, -, \ldots) \cdot 2r_2,$

for any other radius, r_2;

$$\therefore \qquad \frac{(circumference)_r}{(circumference)_{r_2}} = \frac{2r}{2r_2} = \frac{r}{r_2};$$

and if $2r_2 = 2$, and $\therefore$ (373), $(\text{circ.})_{r_2} = 2\pi$, we have

$$\frac{(circumference)_r}{2\pi} = \frac{2r}{2} = r; \text{ i. e.,}$$

The circumference of any circle bears to its diameter a constant ratio. (399)

$$(circumference)_r = \pi \cdot 2r = 3{\cdot}14159265358979\mathrm{93} \cdot 2r.$$

Cor. 1. The arcs of similar sectors are to each other as the radii of the respective circles of which they form like parts. (400)

For, if u, u_2, denote the arcs of similar sectors, or like parts, as the nth, of the circumferences of which the radii are r, r_2, we have

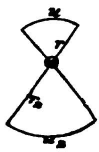


Fig. 65.

$$nu = \pi \cdot 2r,$$

and $\qquad nu_2 = \pi \cdot 2r_2;$

$$\therefore \qquad \frac{u}{u_2} = \frac{r}{r_2}.$$

Eliminating the semicircumference from (398) by aid of (399), there results,

Cor. 2. Area of circle $= \pi r^2$; or, the circle is measured by the square of the radius multiplied into π. (401)

Cor. 3. Circles and the like parts of circles, as similar sectors and segments, are to each other as the squares of their proportional lines—such are the radii, diameters, circumferences, similar arcs and their chords, sines, tangents, secants, and versed-sines. (402)

For let $S + s$ and $S_2 + s_2$ be similar sectors with their equal angles made vertical, s, s_2, the similar segments cut off from the sectors by the chords c, c_2, r, r_2, being the radii. Then $S + s$, $S_2 + s_2$, being like parts of their respective circles (?) if $n(S + s)$ represent the first, $n(S_2 + s_2)$ will represent the second circle; and we shall have (401)

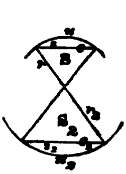


Fig. 66.

$$n(S+s)=\pi r^2,$$

and $$n(S_2+s_2)=\pi r_2^2;$$

$$\therefore\quad \frac{n(S+s)}{n(S_2+s_2)}=\frac{S+s}{S_2+s_2}=\frac{r^2}{r_2^2},\text{ and (160)}=\frac{S}{S_2};$$

$$\therefore\quad \frac{S+s}{S}=\frac{S_2+s_2}{S_2},\text{ or }1+\frac{s}{S}=1+\frac{s_2}{S_2},\ \therefore\ \frac{s}{S}=\frac{s_2}{S_2};$$

$$\therefore\quad \frac{s}{s_2}=\frac{S}{S_2},\ \therefore =\frac{S+s}{S_2+s_2}=\frac{n(S+s)}{n(S_2+s_2)},$$

which shows that the segments, sectors, and circles are proportional. Further, denoting the arcs of the sectors by u, u_2, and $\therefore$ the circumferences by nu, nu_2, we have

$$\frac{nu}{nu_2}=\frac{u}{u_2}=\frac{r}{r_2}=\frac{c}{c_2}=\frac{\sin u}{\sin u_2}=\frac{\tan u}{\tan u_2}=\frac{\sec u}{\sec u_2};$$

$$\therefore\quad \frac{n(S+s)}{n(S_2+s_2)}=\frac{s}{s_2}=\frac{r^2}{r_2^2}=\frac{(2r)^2}{(2r_2)^2}=\frac{(nu)^2}{(nu_2)^2}=\frac{u^2}{u_2^2}=\frac{c^2}{c_2^2}=\frac{\sin^2 u}{\sin^2 u_2}=\ldots.$$

Q. E. D.

PROPOSITION III.

An Incremental Vanishing Arc of any continuous curve, is to be regarded as a straight line. (403)

Fig. 67.

Let z be such an arc, c its chord, and t, t_2, tangents at its extremities, P, P_2, that is, coinciding in direction with z at P, P_2, and terminating in their point of intersection.

We have (387)

$$c=t\cos(t,c)+t_2\cos(t_2,c);$$

but it is obvious from the implied condition of continuity, that as the arc z decreases to its vanishing state and the points P, P_2, approach to coincidence, that the tangents t, t_2, changing their directions by insensible degrees, will come to form one and the same straight line, and the angles, (t,c), (t_2,c), decreasing to zero, their cosines become, $[\cos(t,c)]=1$, $[\cos(t_2,c)]=1$;

Fig. 68.

$$\therefore\quad [c]=[t]+[t_2];$$

and (113), $$z\begin{smallmatrix}<t+t_2\\>c\end{smallmatrix}.$$ Q. E. D.

Scholium. It is obvious that if the curve were other than continuous, no such conclusion as the above would result.

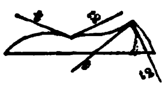


Fig. 69.

PROPOSITION IV.

To find the Derivative of the Segmental Area of any continuous curve referred to rectangular coördinates.

Let Y be segmental area in question, and K its increment; we have (403), (146),

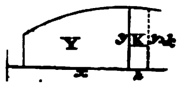


Fig. 70.

$$[K] = [h] \cdot \tfrac{1}{2}(y + y + [k]); \therefore \left[\frac{K}{h}\right] = y + \tfrac{1}{2}[k], \text{ but } k_{h=0} = 0;$$

$$\therefore \quad Y' = \left[\frac{K}{h}\right] = y, \text{ which } = fx; \text{ i. e., } \quad \frac{dA}{dx} = y$$

The derivative of the segmental area of any continuous curve is equal to the ordinate of that curve regarded as a function of the abscissa. [Y′ = y = fx.] (404)

PROPOSITION V.

To find the area of the Ellipse.

Let Y be an elliptical segment embraced by the semi-minor axis and the abscissa x; we have (404), (203), (250),

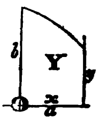


Fig. 71.

$$Y' = y = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} = b\left(1 - \tfrac{1}{2} \cdot \frac{x^2}{a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2} \cdot \frac{x^4}{a^4} - \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{2 \cdot 3} \cdot \frac{x^6}{a^6} +, -, \ldots\right);$$

$$\therefore (246), \quad Y = b\left(x - \tfrac{1}{2} \cdot \frac{x^3}{3a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2} \cdot \frac{x^5}{5a^4} - \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{2 \cdot 3} \cdot \frac{x^7}{7a^6} +, -, \ldots\right), \tag{405}$$

the area sought, no constant being added, since $Y_{x=0} = 0$.

If we make $b = a$, there results

$$Y_c = a\left(x - \tfrac{1}{2} \cdot \frac{x^3}{3a^2} + \frac{\frac{1}{2}(\frac{1}{2} - 1)}{2} \cdot \frac{x^5}{5a^4} - \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{2 \cdot 3} \cdot \frac{x^7}{7a^6} +, -, \ldots\right), \tag{406}$$

the corresponding segment of the circle circumscribing the ellipse;

Fig. 72.

$$\therefore\ Y_e : Y_c :: b : a, \text{ or } Y_e = \frac{b}{a} \cdot Y_c. \quad \text{[Enunciate.]} \tag{407}$$

Making $x = a$ and multiplying by 4, we find

$$(Ellipse)_{a,b} = 4\left(1 - \tfrac{1}{3} \cdot \tfrac{1}{2} + \tfrac{1}{5} \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} - \tfrac{1}{7} \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{2 \cdot 3} +, -, \ldots\right)ab, \tag{408}$$

$$(Circle)_a = 4\left(1 - \tfrac{1}{3} \cdot \tfrac{1}{2} + \tfrac{1}{5} \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} - \tfrac{1}{7} \cdot \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{2 \cdot 3} +, -, \ldots\right)a^2; \tag{409}$$

$\therefore$ *Ellipse* : *Circumscribing Circle* :: $b : a :: 2b : 2a$
:: *Minor Axis* : *Major Axis*. (410)

$$\therefore\ (Ellipse)_{a,b} = \frac{b}{a}(Circumsc.\ Cir.) = \frac{b}{a} \cdot \pi a^2 = ab \cdot \pi. \tag{411}$$

$$(Circle)_b = \pi b^2;$$

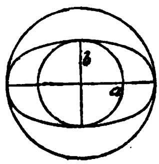


Fig. 73.

$\therefore$ (411), $\quad (Ellipse)_{a,b} : (Circle)_b :: a : b.$ (412)

PROPOSITION VI.

To find the area of the Parabola.

We have (232)

$$Y' = y = 2^{\frac{1}{2}} p^{\frac{1}{2}} x^{\frac{1}{2}};$$

$$\therefore\quad Y = 2^{\frac{1}{2}} p^{\frac{1}{2}} \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \tfrac{2}{3} \cdot 2^{\frac{1}{2}} p^{\frac{1}{2}} x^{\frac{1}{2}} x = \tfrac{2}{3} yx; \text{ i. e.,} \tag{413}$$

The Parabola is two-thirds the circumscribing rectangle. (413$_2$)

Scholium. It is obvious that the quadrature of the circle and ellipse can only be obtained approximately, while that of the parabola is exact.

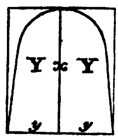


Fig. 74.

PROPOSITION VII.

To find the proximate area of any continuous curve.

Suppose the equation of the curve, referred to rectangular co-ordinates, to be

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots; \tag{414}$$

then $\quad Y' = y = A_0 + A_1x + A_2x^2 + A_3x^3 + \dots;$

$\therefore \quad Y = A_0x + \frac{1}{2}A_1x^2 + \frac{1}{3}A_2x^3 + \frac{1}{4}A_3x^4 + \dots + constant,$

but, if the area Y become $= 0$ when $x = 0$, which condition is always admissible, since the origin may be taken at pleasure, we have

$$Y = A_0x + \frac{1}{2}A_1x^2 + \frac{1}{3}A_2x^3 + \frac{1}{4}A_3x^4 + \dots . \tag{415}$$

But, since three points, P_0, P_1, P_2, determine with considerable accuracy a curve of moderate extent, we will take the foot of the first ordinate, y_0, as the origin, and the abscissas, $x_1 = h$, $x_2 = 2h$, so that the corresponding ordinates, y_0, y_1, y_2, shall be equally distant from each other; $\therefore$ making $x = 0, h, 2h$, we have (414)

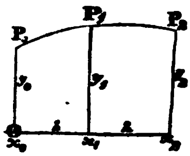


Fig. 75.

$$y_0 = A_0,$$
$$y_1 = A_0 + A_1 \cdot h + A_2 \cdot h^2,$$
$$y_2 = A_0 + A_1 \cdot 2h + A_2 \cdot 4h^2;$$

$\therefore \qquad y_1 - y_0 = A_1 \cdot h + A_2 \cdot h^2,$

and $\qquad y_2 - y_1 = A_1 \cdot h + A_2 \cdot 3h^2;$

$\therefore \qquad y_2 - 2y_1 + y_0 = A_2 \cdot 2h^2;$

but when $x = 2h$, we have (415)

$$Y = A_0 \cdot 2h + \tfrac{1}{2}A_1 \cdot 4h^2 + \tfrac{1}{3}A_2 \cdot 8h^3 = 2h(A_0 + A_1h + 1\tfrac{1}{3}A_2 \cdot h^2);$$

$\therefore \qquad Y = 2h[y_1 + \frac{1}{3} \cdot \frac{1}{2}(y_2 - 2y_1 + y_0)],$

or $$Y = \tfrac{2}{3}h(\tfrac{1}{2}y_0 + 2y_1 + \tfrac{1}{2}y_2). \tag{416}$$

If we continue the same method of admeasurement and notation, we have

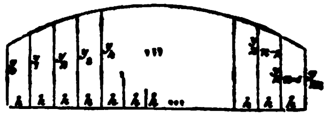


Fig. 76.

$$Y_2 = \tfrac{2}{3}h(\tfrac{1}{2}y_0 + 2y_1 + \tfrac{1}{2}y_2),$$
$$Y_4 = \tfrac{2}{3}h(\tfrac{1}{2}y_2 + 2y_3 + \tfrac{1}{2}y_4),$$
$$Y_6 = \tfrac{2}{3}h(\tfrac{1}{2}y_4 + 2y_5 + \tfrac{1}{2}y_6),$$
&c. &c.
$$Y_{2n} = \tfrac{2}{3}h(\tfrac{1}{2}y_{2n-2} + 2y_{2n-1} + \tfrac{1}{2}y_{2n});$$

$$\therefore\ Y_2+Y_4+Y_6+\ldots+Y_{2n}=\tfrac{4}{3}h(\tfrac{1}{2}y_0+2y_1+y_2+2y_3+y_4 \\ +2y_5+\ldots+y_{2n-2}+2y_{2n-1}+\tfrac{1}{2}y_{2n}). \qquad (417)$$

This beautiful and useful theorem is due to Simpson. The student should enunciate it in common language.

EXERCISES.

1°. Required the diameter of a circle having ten linear chains in circumference to every square chain in area.

2°. A square plate of silver, 3 inches on the side, is worth \$4. What is the value of the greatest circle that can be cut from it?

3°. Had the plate in 2° been an equilateral triangle, what would have been its value?

4°. The two sides including the right angle of a right angled triangle, are three and four rods; what is the area of the circumscribing circle?

5°. Determine a circle circumscribing an isosceles triangle, the two equal sides of which, including an angle of 36°, are 15·15 chs. each?

6°. The equal sides of an isosceles triangle embrace an angle of 47·3°, and the area of the inscribed circle is one acre. Determine the triangle.

7°. A circular plate of brass, 20 inches in diameter, is worth \$3·75. What is the value of the three greatest and equal circles that can be cut from it?

8°. Required the area of a circular segment embraced by an arc and its chord, the length of which is 5·87 chs., and the breadth 1·35?

9°. The dimensions of an elliptical fish-pond are 10 and 15 rods. What is its area?

10°. The ordinate of a parabolic segment is 3 chains, and the corresponding abscissa 7 chains. Required the area.

11°. Required the area of an elliptical segment embraced between the semiminor axis and an ordinate $=5·657$ chains, and having a breadth of one chain.

12°. Wishing to ascertain the cross section of a river 100 yards from water's edge to water's edge, I take soundings every 10 yards, and find them to be in yards:

$y_0=0$, $y_1=12$, $y_2=20·1$, $y_3=25·3$, $y_4=28·4$, $y_5=29·9$, $y_6=29·3$, $y_7=26·1$, $y_8=20·9$, $y_9=12·8$, $y_{10}=0$.

BOOK THIRD.

SURVEYING.

SECTION FIRST.

Description and Use of Instruments.

To determine the boundaries of lands, to delineate them in maps, and to compute their areas, constitute the Art of Surveying.

The instruments employed in determining the boundaries of lands are, the *Chain*, for measuring the lengths of lines, the *Surveyor's Cross* to determine right angles, and the *Azimuth Compass*, or the *Theodolite*, for fixing the inclinations of lines to each other and to the meridian.

The Chain.

Gunter's chain, as has already been observed, is 4 rods or 66 feet in length, and centesimally divided by a hundred links, each, consequently, equal to 7·92 inches.

It is a maxim in land surveying that *every instrument, whether for measuring lines or angles, must be used in a horizontal position;* for it is the base, or the *projection* of the field upon the same horizontal plane, that is desired.

The projections of the bounding lines are usually obtained by carrying the chain in a horizontal position, as represented in *fig.* 77, where 1, 2, 3, ..., are the successive positions of the chain.

Fig. 77.

When the inclination of the ground is too great to admit of a whole chain, a half or a quarter may be taken, and in all cases the proper position of the elevated extremity should be determined by a plumb line or by the dropping of a stone.

The chief points to be attended to in chaining are, 1°, to keep the chain in a horizontal position; 2°, to avoid straying from the line; 3°, to record without error the number of chains.

The first condition will be secured by the surveyor supporting, when necessary, the middle of the chain, and directing the elevation or depression of its extremities.

The second condition will be readily attained by the foreman fixing his attention upon the flag-staff or object of sight, thus drawing the chain constantly in line. His march will be corrected by the hindman, who will cry out "RIGHT!" "LEFT!" as the occasion may require.

Error in record will be guarded against by employing ten iron pins which should be turned at the top into a small ring, and tied with a piece of red flannel, the better to be seen. The foreman takes the ten pins and draws out the chain, the hindman, as it is near being stretched, cries "DOWN!" when the foreman, giving the chain a wave to bring its parts in line, pulls it tight and puts down a pin. Marching on he repeats the same operation, until, coming out empty-handed, he puts his foot upon the extremity of the chain to secure it in place, and cries "TALLY ONE!" and the hindman responds "*tally one!*" that the number may be fixed in the memory, also recording it in some way, as by a notch in a stick or a pebble put in the pocket, if thought necessary. He then, quitting the hind end of the chain, marches up to the foreman, who counts the pins to assure himself of the reception of the ten; when, stretching on, the second tally is executed like the first. A field may be surveyed by the chain alone, as illustrated by the subjoined

Field Notes.

Contour. AB = 2·37, BC=4·67, CD=5·00, DE=4·98, EA=3·67,
Diagonals. BD = 4·83, BE = 5·25 chs.

Required the angles of the pentagon.

The question naturally arises: ought we not to measure the inclined plane rather than its horizontal projection, since the surface of the former exceeds that of the latter? The answer to this question must be given in the negative, and for two reasons; 1°, a uniformity in surveying different lands is desirable, that they may be the more readily compared with each other, and it is obvious that this uniformity can be attained only by reducing them to their horizontal projections; 2°, the real value of a field cannot exceed

that of its horizontal projection, since no more soil will rest on an inclined plane than on its horizontal projection, and the same number of plants will stand upon the one as on the other.

For let *abcd* represent a vertical section of the soil of an inclined plane; it is only equal to its horizontal projection *efgh*, the vertical depth *ad* being supposed equal to *eh*; and trees will grow as thick together on the inclined as on the horizontal plane.

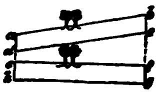


Fig. 78.

The Surveyor's Cross.

The Cross has been already described and the method of using it in surveying pointed out. [See Book Second, Section First.] The student may now employ the chain and cross to survey a small field, and then compute its angles.

The Compass.

The Surveyor's or Azimuth Compass consists of a horizontal circle to which are attached sight-vanes, and a magnetic needle delicately balanced on its centre, by which the vanes may be directed to any point of the horizon so as to determine the inclination of lines to the magnetic meridian, and, consequently, to each other. The degrees, marked on the limb of the instrument, are numbered from the north and south points, N. and S., both ways to the east and west, designated by the opposite letters, W. and E., for a reason that will presently appear.

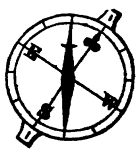


Fig. 79.

To use the compass, set it firmly upon its staff (better and usually a tripod), furnished with a ball and socket joint, capable of being loosened or tightened at pleasure, by the aid of which and two spirit levels, placed at right angles to each other on the face of the instrument, the limb is to be brought into a horizontal position. When this is effected will be known by the bubbles remaining in the middle of the levels while the instrument is made to revolve on its axis. The needle is now to be let carefully down upon its pivot by a screw in the under side. See that it plays with its points just skimming along the graduated edge of the limb. Turn the vanes into the required direction by sighting at a staff wound with a red flag, and held vertically in line by an assistant stationed at a suitable distance. Observe if the needle settles with

a free motion, describing nearly equal arcs, slowly decreasing, on each side of a definite point, and if it finally rests at that point. Should there be any doubt as to this, the needle must be agitated, either by the attraction of a knife, or by gently tapping the tripod with the fingers, and be permitted to settle a second time. To insure a correct position of the needle is the principal difficulty in operating with the compass.

In order to prevent mistakes, that sight should be turned toward the flag staff which will bring the north end of the needle into the part of the compass marked N., or with the fleur de luce. The bearing will then be read off by the forward end of the needle, using the letters it stands between, as in the figure, N. 30° E., if the course be northerly, or S. 30° W., if in the opposite direction; both ends, however, are to be observed in estimating the amount of the angle. Back sights should be taken at each station in order to verify the bearings.

The *Vernier* or *Nonius* is a slip of metal, fitted to slide upon the graduated limb of an instrument, and to serve the purpose of an extended and impracticable subdivision.

If x denote the value of a division on the vernier, of which n cover $n \pm 1$ divisions of the instrument, we have

$$nx = n \pm 1,$$

$$x = 1 \pm \frac{1}{n};$$

$$x - 1 = \pm \frac{1}{n},$$

$$2x - 2 = \pm \frac{2}{n},$$

$$3x - 3 = \pm \frac{3}{n}, \text{ \&c., \&c.};$$

from which it appears that the distance, $x - 1$, of the first division of the vernier from the first division of the instrument, will be an nth part of the unit of graduation—the distance, $2x - 2$, of the second division of the vernier from the second division of the limb will be two $n$ths of the unit of graduation, and so on; so that, by sliding the vernier along the limb, we shall be enabled to measure spaces to the nth part of the smallest divisions of the instrument. Thus, if a scale have a division of 10ths, and 11 of these be covered by a

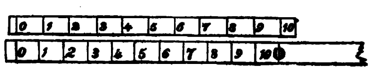


Fig. 80.

vernier of 10 divisions, we have $n = 10$, and $\therefore\ x - 1 = \frac{1}{10}$ of $\frac{1}{10}$, $2x - 2 = \frac{2}{10}$ of $\frac{1}{10}$, ...; so that, by sliding the vernier along the scale to make the points, 1, 1; 2, 2; 3, 3, ..., agree in succession, there results the measures ·01; ·02; ·03, ... In like manner, if 30 divisions of the vernier attached to the compass, cover 29 half-degree divisions of the instrument, we shall be enabled to measure angles to the minute. In some compasses the vernier is attached to the extremity of the needle, and, being carried along by it, the degrees and parts of a degree are read by simple inspection; in other instruments it is on the outside of the limb, and fastened by a clamp-screw from below, which must be loosened, when the vernier will be driven by a tangent screw adapted to the purpose.

The Theodolite.

The Theodolite consists essentially of a vertical and horizontal circle, for the purpose of measuring angles in altitude and azimuth. [*Frontispiece.*] It has, like the compass, spirit levels, by the aid of which and screws, the azimuth circle may be brought into a horizontal position. When this is accurately accomplished, the theodolite is ready to measure any horizontal angle having the angular point at its centre, provided all the parts of the instrument have been carefully adjusted. To this end, direct both the upper and lower telescopes (the first attached to the vertical circle, the second to the axis below the horizontal) to the same mark situated at a distance in one of the sides of the angle. Observe the position of the vernier upon the limb of the azimuth circle, reading the degrees and parts of a degree by one or more microscopes, fitted to this end—unclamp the upper telescope and direct it to a mark in the second side of the angle, clamping and finishing the motion by aid of the tangent screw. Observe, by the lower telescope, whether the azimuth circle has suffered any displacement by the motion required in making the second observation; if no such derangement has happened, the difference of the first and second readings will be the measure of the angle in question. In order to secure greater accuracy, the axis of the azimuth circle may be unclamped, the upper telescope brought back to the first mark, carrying the azimuth circle along with it—the azimuth again clamped, and the angle measured a second time. This operation repeated as often as desirable, the whole amount of arc passed over divided by the

number of observations, will give the angle required with a corresponding degree of exactness. We scarcely need say that the eye-glass must be drawn out or pushed in till the cross wires, indicating the line of sight—line of *collimation*, as it is called—shall be seen distinctly, and that the object glass is to be moved in like manner till the mark becomes well defined. In a similar way, a vertical angle will be measured by the vertical circle, having previously brought the telescope to a horizontal position and observed the reading, which should then be zero. When this horizontality shall be accomplished, will be determined by a level attached to the upper telescope, or, if no such level exist, by those already mentioned.

There are several adjustments, either permanently fixed by the instrument maker, or, for the execution of which, he furnishes means in screws and parts capable of being detached from each other.

There are five lines that should be respectively perpendicular to each other, viz., the vertical axis, or axis of the azimuth circle, the horizontal axis, or axis of the vertical circle, the horizontal line, or line of collimation when the vertical circle indicates zero, the vertical wire, and the horizontal wire—or

$$\angle\,(A_h, A_v) = (H, A_v) = (H, A_h) = (h, A_v) = (v, A_h) = (h, H) = (v, H) = 90^\circ\,;$$

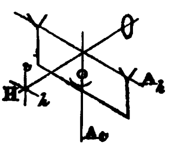


Fig. 81.

also the circles should be perpendicular to their axes. The method of testing these adjustments consists principally in reversing the lines (73); for which purpose the telescope and horizontal axis, one or both, may be lifted from their Ys or supports, the object and eyeglasses change places, or the vernier plate carrying the vertical circle and the telescope, revolved 180°. Also the wires, *h*, *v*, which form by their intersection the line of collimation, being attached to a ring, may be moved to the right or left, elevated or depressed by screws from without.

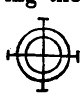


Fig. 81a.

If we direct the line H to a distant and well-defined mark, and, when the telescope is reversed, find the sight upon the same mark, we may be assured that the line of collimation, H, is perpendicular to the horizontal axis A_h. This adjustment perfected, the horizontal axis, A_h, will be perpendicular to the vertical, A_v, when, passing the line of collimation,

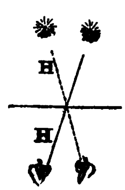


Fig. 81b.

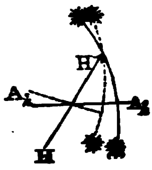


Fig. 81$_4$.

H, through two marks having a considerable angular distance, and turning the vernier plate 180°, H continues to pass through the same points.

The vertical wire v will be in a plane perpendicular to the horizontal axis A_h, when, moving the vertical circle backward and forward, a distant and well-defined mark continues accurately on v. In order to know whether the horizontal wire, h, be in its proper position, or if the line of collimation, H, when the vertical circle indicates zero, be perpendicular to the vertical axis, A_v, it is only necessary to reverse the telescope, and that the wire h is in a plane parallel to the azimuth circle, will be determined by a backward and forward motion of the vernier plate. When these adjustments shall have been perfected, by often repeating them one after another in a different order, whether the levels are parallel to the plane of the azimuth circle will be known by leveling this circle, making the vertical circle indicate zero, if its telescope have a level attached to it, revolving the vernier plate and seeing if the bubbles continue in the middle. Whether the level which may be attached to the telescope be perpendicular to the axis A_h, will be known by bringing A_h over two of the leveling screws, and then, by aid of these screws flinging A_h out of level, or by revolving the telescope in its Ys, if it be capable of such a motion.

If, on any occasion, it be desired to make the vertical circle coincide with the greatest possible accuracy with a vertical plane, we may suspend a plumb line before the telescope and observe when the line of collimation traces this line.

The Variation of the Magnetic Needle may be conveniently determined with the theodolite by the process of equal altitudes.

Let the magnetic bearing of the sun before noon at a determinate altitude be $e°$, and at the same altitude after noon $f°$, and suppose x = variation; then will the angular distances of the sun from the true meridian be, before noon, $e \pm x$, afternoon, $f \mp x$, but these distances are equal,

$$\therefore \qquad e \pm x = f \mp x, \text{ and } x = \frac{f-e}{\pm 2}.$$

The equal altitudes may be taken at any corresponding hours before and after noon, and in any season of the year, but the most favorable time is about the 21st of June, when the sun, being near the summer solstice, will change his declination but a few seconds

between the observations; and, if these determinations be made several successive days, both before and after the 21st, the errors, lying in opposite directions, will tend to balance each other; thus, suppose the values of x

for June	18, 19, 20, 21, 22, 23, 24,
to be	x_1, x_2, x_3, x_4, x_5, x_6, x_7;

then $$x = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7}{7}.$$

The observations may be made several times during the same day, but the best hours will be when the sun is in a position nearly east or west.

The Art of Leveling consists in finding the difference of elevation between two places: and is not only necessary in the construction of railroads, aqueducts, and canals, but is useful for laying the foundations of edifices, and for other like purposes. The operation may be readily executed by aid of the theodolite, or, more conveniently, with the *Leveling Instrument* made expressly for the purpose, and consisting of a telescope, level, and tripod, being in all respects similar to the theodolite, except in not possessing the graduated circles. The instrument is to be firmly planted midway between two stations, situated at a convenient distance, and its telescope made to revolve accurately in the horizontal plane, when the depressions of the first and second stations are to be noted by sighting at a graduated staff held vertically on these two points in succession. In like manner the second station is to be compared with a third, the third with a fourth, and so on, to the last.

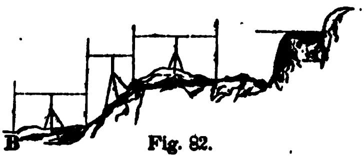
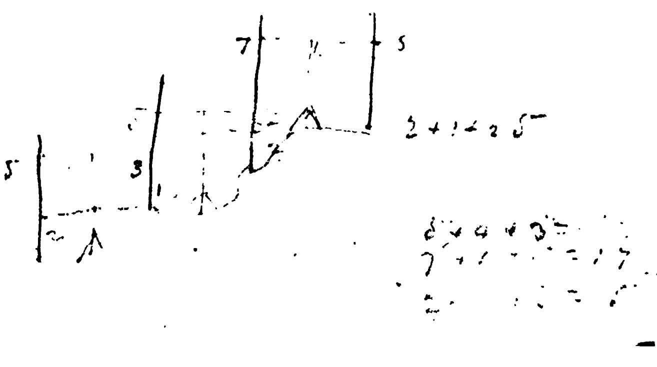


Fig. 82.

For an example, suppose it required to find the elevation of the fountain, A, above a dwelling house at B, from the following notes, in which the back altitudes are marked + and the forward ones —.



+	−
5·23	2·78
6·06	3·17
5·82	1·19
6·75	0·92
4·33	4·87
2·15	7·12
30·34	20·05
20·05	
10·29	

Ans. 10·29 feet elevation.

SECTION SECOND.

Plotting.

PROBLEM I.

Through a given point to draw a line perpendicular to a given line.

Join the given point, P, and any convenient point, Q, of the given line; with the middle, O, of the line PQ as a centre, describe the semicircumference PP′Q, cutting the given line in P′; the line drawn through P, P′, will be the perpendicular required. Why?

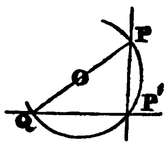


Fig. 83.

If the given point be P′ in the given line, set one foot in P′ and the other in any convenient point, O, out of the line, describe a circle and draw the diameter QOP; PP′ will be the perpendicular required.

Other methods of drawing perpendiculars may be employed, as indicated in the figures, but the "Right-angle" is preferable in practice.

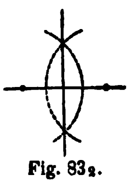


Fig. 83_2.

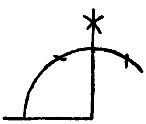


Fig. 83_3.

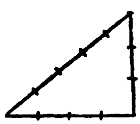


Fig. 83_4.

PROBLEM II.

Through a given point to draw a line parallel to a given line.

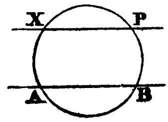


Fig. 84.

Set one foot of the dividers in the given point, P, and with any convenient centre describe the circumference PBAX, cutting the given line in A and B; take the chord PB and apply it from A upon the circumference at X; PX will be the parallel required. (Why?) Do the same thing with the Right-angle and Straightedge.

PROBLEM III.

From a definite point in a given line to make an angle equal to a given angle.

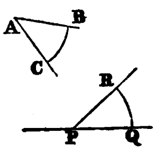


Fig. 85.

Around the given angle, A, with any convenient radius, AB, describe the arc BC; around the given point, P, with the same radius, describe the arc QR, Q being a point in the given line; apply the chord BC in QR; QPR will be the required angle.

PROBLEM IV.

To construct a triangle, having given its three sides.

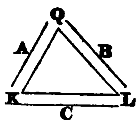


Fig. 86.

Draw an indefinite line, KL, in the required position, and apply one of the given sides, C, from K to L; with the other sides, A, B, as radii, describe around the centres, K, L, arcs intersecting in Q; KQL will be the triangle required.

Scholium I. This problem enables us to plot a field when it is surveyed by the chain, that is, when its diagonals are known, either by actual measurement or by computation. The student will find it a profitable exercise to plot the pentagon given in the preceding section under the chain.

Scholium II. The preceding *Graphical Problems* give us the six following *Problems of Construction*, whereby any geometrical problem, solved algebraically, may be executed in a geometrical way.

PROBLEM V.

To construct the sum of two lines,

$$x = a + b.$$

Lay off b upon the production of a.

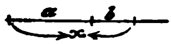


Fig. 87.

PROBLEM VI.

To construct the difference of two lines,

$$x = a - b.$$

Lay off b from one extremity of a towards the other, as from the right hand towards the left in the figure; the remainder will be $= x$.

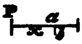


Fig. 88.

We observe that if b exceed a, x will be drawn to the *left* instead of to the *right* (180), and that x will be *minus* instead of *plus* in the equation $x = a - b$.

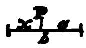


Fig. 88₂.

PROBLEM VII.

To construct the square root of the sum of the squares of two lines,

$$x = (a^2 + b^2)^{\frac{1}{2}}.$$

Make a and b the sides of a right angled triangle; x will be the hypothenuse.

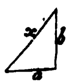


Fig. 89.

PROBLEM VIII.

To construct the square root of the difference of the squares of two lines.

$$x = (a^2 - b^2)^{\frac{1}{2}}.$$

On the greater line, a, describe a semicircle, and from the extremity of a lay off the chord b; x will be the chord joining the extremities of a and b.

Fig. 90.

PROBLEM IX.

To construct a fourth proportional,

$$a : b :: c : x, \text{ or } ax = bc, \text{ or } x = \frac{bc}{a}.$$

Inscribe the chord $b+c$ in a circle of suitable radius, then with a in the dividers and one foot in the junction of b, c, cut the circumference; the intersection will give the position of a, such that its production, intercepted by the opposite part of the circumference, will be x.

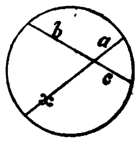


Fig. 91.

This problem may also be solved as indicated by fig. 23_2.

Scholium. If c be made equal to b, or if $a : b :: b : x$, the problem reduces to that of constructing a *Third Proportional.*

PROBLEM X.

To construct a mean proportional.

$$b : x :: x : c, \text{ or } x = (bc)^{\frac{1}{2}}.$$

In the above, make $b+c$ a diameter, and x perpendicular to $b+c$; then will a of fig. 91 become $= x$ in fig. 92.

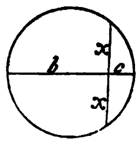


Fig. 92.

Scholium. This problem solves also VIII., since

$$x = (a^2 - b^2)^{\frac{1}{2}} = [(a+b)\ (a-b)]^{\frac{1}{2}},$$

where x is a mean proportional between $a+b$ and $a-b$; again, the eighth may be advantageously employed in executing the present—seeing that

$$x = (bc)^{\frac{1}{2}} = \left[\left(\frac{b+c}{2}\right)^2 - \left(\frac{b-c}{2}\right)^2\right]^{\frac{1}{2}};$$

for b is the sum of $\quad \dfrac{b+c}{2}$ and $\dfrac{b-c}{2}$,

and c the difference of $\quad \dfrac{b+c}{2}$ and $\dfrac{b-c}{2}$.

Scholium. It is to be observed that an equation, in order to be capable of construction, must be homogeneous (370); and, consequently, that it must not embrace any unlike quantities, such as lines and surfaces. Further, all geometrical quantities are to be expressed in lines, as these are the only magnitudes which we can measure directly; thus, a surface S by a square, a^2, whose side is a; a solid V by a cube, a^3, having the side a; a ratio r by two lines, m, n, $r = \dfrac{m}{n}$. Such an equation as $x^2 + 2ax = b$ cannot be constructed, not being homogeneous.

As an exercise, let it be required to construct the general quadratic equation

$$x^2 + 2ax = bc,$$

where a, b, c, are known lines.

We shall find $\qquad x = -a \pm [a^2 + bc]^{\frac{1}{2}},$

which may be put under the form

$$x = -a \pm \left[a\left(a + \frac{bc}{a}\right)\right]^{\frac{1}{2}};$$

$\therefore$ 1°, construct $\frac{bc}{a}$ by Prob. IX.;

2°, construct $a + \frac{bc}{a}$ by Prob. V.;

3°, construct $\left[a\left(a + \frac{bc}{a}\right)\right]^{\frac{1}{2}}$ by Prob. X.;

4°, construct $x = -a \pm \left[a\left(a + \frac{bc}{a}\right)\right]^{\frac{1}{2}}$ by Probs. V. and VI.

As a second exercise, let it be required to divide a line into *Extreme* and *Mean Ratio*, i. e., so that the whole line, AB, shall be to its greater part, AX, as this greater part, AX, is to the less, XB.

If the parts be denoted by $a + x$ and $a - x$, we shall find

$$x = -2a \pm [a(a+a+a+a+a)]^{\frac{1}{2}}, \text{ or } = -2a \pm [2a \cdot 2\tfrac{1}{2}a]^{\frac{1}{2}}.$$

The student will find it a useful exercise to construct the problems which he has already solved, as those at the end of Section Third, Book Second. He should also endeavor to combine, as much as possible, elegance and simplicity in the arrangement and execution of the several parts of a complicated construction. Theorems other than those involved in the foregoing problems, as (182), (185), may frequently be advantageously employed.

PROBLEM XI.

To divide the circumference of a circle into equal parts, as degrees.

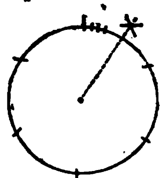


Fig. 93.

First Method. With the required radius describe a circumference, to which the dividers, unaltered, will apply six times, since the chord of 60° is = to radius. Next bisect the arc of 60° thus found either by intersecting arcs described from its extremities and a line drawn to the centre, or by trial;

in like manner bisect this arc of 30°, which must be divided into three parts by trial; and, lastly, the arc of 5° is to be divided by the same method.

Second Method. Having graduated a quadrant of large radius, as above, transfer the chords to a single line. This line, engraved upon a ruler, is denominated the *Line of Chords.* To make use of it, we have only to describe a circumference with the chord of 60° as radius, and then to this circumference apply the chord of the required arc, also taken from the Line of Chords.

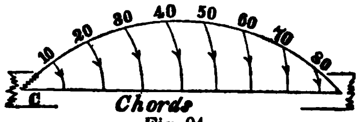


Fig. 94.

Third Method. Lay off the arc from a scale of equal parts, by aid of the following

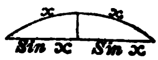


Fig. 95.

Table of Chords. Chord $2x = 2\sin x$, $r = 1$.

	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
0°	0·000	0·174	0·347	0·518	0·684	0·845	1·000	1·147	1·286	1·414
1°	0·018	0·192	0·365	0·535	0·700	0·861	1·015	1·161	1·299	1·426
2°	0·035	0·209	0·382	0·551	0·717	0·877	1·030	1·176	1·312	1·439
3°	0·052	0·226	0·399	0·568	0·733	0·892	1·045	1·190	1·325	1·451
4°	0·070	0·244	0·416	0·585	0·749	0·908	1·060	1·204	1·338	1·463
5°	0·087	0·261	0·433	0·601	0·765	0·924	1·075	1·218	1·351	1·475
6°	0·105	0·278	0·450	0·618	0·782	0·939	1·089	1·231	1·364	1·486
7°	0·122	0·296	0·467	0·635	0·798	0·954	1·104	1·245	1·377	1·498
8°	0·140	0·313	0·484	0·651	0·814	0·970	1·118	1·259	1·389	1·509
9°	0·157	0·330	0·501	0·668	0·829	0·984	1·133	1·272	1·402	1·521

This method will be found particularly convenient for mechanics who have at hand a ruler graduated into inches and 8ths, or, better, 10ths. Thus, with a radius of 20 inches, the chord of 57° 45′ is

$$20(0{\cdot}954 + \tfrac{3}{4} \cdot 16) = 19{\cdot}32 \text{ inches.}$$

Fourth Method. Find how many times the chord of 1° is contained in radius. We have

$$\text{chord } 1^\circ = 2\sin\tfrac{1}{2}^\circ\,;$$

$$\therefore \quad \log.\ \text{chord } 1^\circ = 0{\cdot}3010300$$
$$+\,\bar{3}{\cdot}9408419$$
$$= \bar{2}{\cdot}2418719\,;$$

$$\therefore \qquad \log. \frac{rad. [=1]}{chord\ 1^\circ} = 1{\cdot}7581281,$$

and $$radius = 57{\cdot}2965 \text{ chord } 1^\circ,$$

or $$= 57{\cdot}3 \text{ chord } 1^\circ \text{ nearly.}$$

Hence a circumference described with a radius of 57·3 taken from a scale of equal parts, will be readily subdivided, the chord of a degree being one of these parts.

Circles, semicircles, and short rulers, made of brass or ivory, and graduated on their edges, are also employed for laying down angles. A particular description of these instruments, as well as of others, usually contained in a surveyor's case, such as steel points and pens for drawing blank lines and describing circumferences in ink, is unnecessary, as their use will be obvious on inspection.* But the first method, which requires only a pair of dividers, will be found quite as accurate as any, and perhaps as expeditious. We scarcely need say that all exact drawings should be made with a steel point and afterward inked so far as required. A very convenient and good pen for describing circumferences may be had by thrusting one leg of the dividers through a common pen, having cut for the purpose a gash downward in front, just above the clear part, and another upward on the back side, a little above the point.

PROBLEM XII.

To plot a field, having given the lengths and bearings of its sides.

As an example, let it be required to plot the first of the following *Field Notes.*

Draw a vertical line through the middle of the paper for a meridian, the top of which is to be regarded as north and the bottom as south, then the right hand will be marked E., and the left W.

* A pair of dividers having three legs will be found very convenient in copying. See Lerebours' Catalogue.

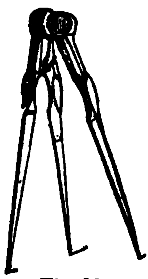


Fig 95.

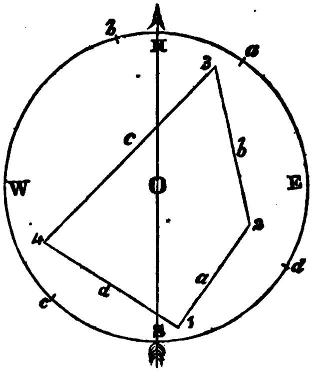


Fig. 96.

Next, with one foot of the dividers centrally situated in the meridian NS, describe a circle as large as the paper will conveniently admit of, when the bearings are to be laid off, by the preceding problem, upon the circumference from the north and south points, Na = 36·5° towards E., Nb = 15·3° towards W., Sc = 46°, towards W. Laying the perpendicular of the rightangle upon the centre, O, and the point of the circumference, a, and applying the straightedge to its base, transfer the line Oa to a convenient position, a, and, having constructed or chosen a scale of equal parts, lay off a = 15·75 chains from the first station, 1, to the second, 2; this done, b is to be drawn parallel to Ob through the second station, 2, and measured off from the same scale, when, the third station, 3, being thus determined, the side c is to be laid down in like manner; and lastly d.

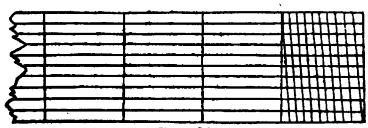


Fig. 97.

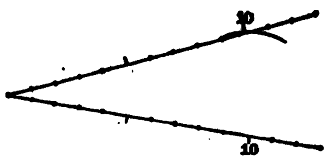


Fig. 98.

The *Diagonal Scale* is convenient for laying down lines, also for graduating the circumference, either by the third or fourth method. The *Sector is* likewise frequently employed, on account of the facility with which its unit is varied to suit the dimensions required in a given plot, simply by causing its branches to approach or recede like the parts of a common jointed rule. Thus, if the sector be opened so that from 10 on one side to 10 on the other shall = 1 inch, then from 20 to 20 will = 2 inches, from 21 to 21 will = 2·1, and so on. A sectoral scale may be constructed on paper for the occasion, and to any required unit, say 10 to the half inch, by first drawing an arc with a radius = ½ inch, then running off from the centre, with any convenient opening of the dividers, a line of 10 equal parts and from the extremity drawing a secant to the arc, to which the same parts may be applied.

A very small pair of dividers going with a screw, or, simply, a fork cut in metal by a file, or even in hard wood with a knife, will be found convenient.

PROBLEM XIII

To reduce or enlarge a plot.

From any point conveniently situated, draw lines passing through the several stations—then draw, in succession and terminating in these lines, parallels to the sides of the plot.

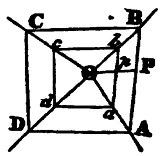


Fig. 99.

The student will exercise himself in plotting all the following fields, measuring the last sides and their bearings.

Field Notes.

I.

a	N 36‘5° E	15‘75 chs.
b	N 15‘3° W	20‘00
c	S 46° W	30‘25
d	S 58‘24° E	20‘781

II.

a	N 80° W	45‘53 chs.
b	N 2° W	65‘23
c	N 86° E	57‘86
d	S 32° E	50‘00
e	S 46° 53′ 23″ W	50‘82

III.

a	S 48‘4° W	20 chs.
b	S 78‘5° W	15
c	N 31‘3° W	25
d	N 35‘6° E	33
e	S 69° E	23
f	S 4° 43′ 54″ E	23‘764

IV.

a	N 69·0° W	23 chs.
b	N 28·4° W	17
c	N 31·6° E	11
d	N 56·8° E	15
e	S 67·1° E	16
f	S 19·0° E	22
g		

V.

In order to determine a line rendered inaccessible by intervening declivities, I trace the line ABCDEFG through a neighboring ravine, and find AB, N 23° E, 25 chs.; BC, N, 31 chs.; CD, N 5° W, 10 chs.; DE, N 12° W, 15 chs.; EF, N 10° E, 35 chs., to the top of the ravine, thence to the second extremity, G, of the required line, S 45° W, 51·87 chs. Required, AG.

VI.

Find where the meridian, passing through the middle point of the side *a* of IV., will intersect the opposite part of the perimeter.

Ans. At a point in *e* distant 55 links from the extremity of *d*.

SECTION THIRD.

Computation of Areas.

PROPOSITION I.

If the sides of any polygon be projected on the same line, the sum of these projections, taken in order with their proper signs, will obviously be equal to zero. (418)

Thus if the sides AB, BC, CD, DE, EA, of the polygon ABCDEA, be projected on the line LL, in ab, bc, cd, de, ea, we evidently have

$$ab + bc + cd + (-de) + (-ea) = 0.$$

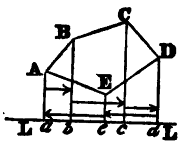


Fig. 100.

And, generally, if we regard the perimeter as described by a point revolving about the polygon

in a constant direction, as ABCDEA, then will the projection of this point upon any line given in position, as LL, describe, by its motion, the projection of the perimeter, which projection, increasing and diminishing, will obviously become nothing when the revolving point returns to its first position, as A.

Scholium. The principle enunciated in (180), applied to this problem, serves to distinguish the plus and minus projections, which will be found to be measured to the right or left, corresponding with the motion of the revolving point. Thus, as the point revolves through ABC, the projection, measured towards *d*, decreases, becoming $= 0$, when the point arrives at D, and then reappears, measured in the contrary direction, as the point returns through DEA.

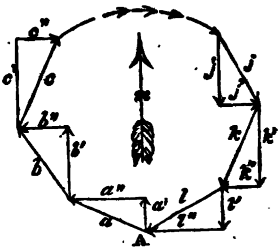


Fig. 101.

It follows that if a, b, c, ..., j, k, l, denote the sides of a polygon taken in order, a', b', c', ..., j', k', l', their projections north and south, a'', b'', c'', ..., j'', k'', l'', the corresponding projections east and west, and m a meridian line; then will the sum of the meridional projections, taken with their proper signs, be $= 0$, as also the sum of the projections at right angles to the meridian; and we shall find

$$\left.\begin{aligned} & a' + b' + c' + \ldots + j' + k' + l' = 0, \\ \text{or}\quad & a\cos(a,m) + b\cos(b,m) + c\cos(c,m) + \ldots \\ & \qquad + j\cos(j,m) + k\cos(k,m) + l\cos(l,m) = 0, \\ \text{since}\quad & a' = a\cos(a,a') = a\cos(a,m),\ b' = \cos(b,m),\ \ldots, \\ & \qquad l' = l\cos(l,m); \end{aligned}\right\} \quad (419)$$

$$\left.\begin{aligned} \text{and}\quad & a'' + b'' + c'' + \ldots + j'' + k'' + l'' = 0, \\ \text{or}\quad & a\sin(a,m) + b\sin(b,m) + c\sin(c,m) + \ldots \\ & \qquad + j\sin(j,m) + k\sin(k,m) + l\sin(l,m) = 0, \\ \text{since}\quad & a'' = a\cos(a,a'') = a\sin(a,a') = a\sin(a,m),\ \ldots, \\ & \qquad l'' = l\sin.(l,m); \end{aligned}\right\} \quad (420)$$

$$\left.\begin{aligned} \therefore\quad & \frac{a\sin(a,m) + b\sin(b,m) + \ldots + k\sin(k,m)}{a\cos(a,m) + b\cos(b,m) + \ldots + k\cos(k,m)} \\ & \qquad = \frac{-\,l\sin(l,m)}{-\,l\cos(l,m)} = \tan(l,m), \end{aligned}\right\} \quad (421)$$

by which (l,m) becomes known;

then (420) $l = \dfrac{a\sin(a,m) + b\sin(b,m) + \ldots + k\sin(k,m)}{-\sin(l,m)}$; (422)

and the last side, l, is completely determined—that is, its bearing and length are drawn from the bearings and lengths of the other sides. Enunciate the above forms. How is the denominator of (421) formed from the numerator?

Cor. 1. If the sides of a polygon, except one, vary in such way as to preserve their mutual ratios and inclinations constant, then will this excepted side bear to any one of the others a ratio and inclination also constant.* (423)

For, denoting the constant ratios which $b, c, \ldots, k$, bear to a by $r_b, r_c, \ldots, r_k$, or putting

$$b = r_b a, \quad c = r_c a, \quad \ldots, \quad k = r_k a,$$

which is in accordance with the condition that any one of the sides, $a, b, c, \ldots, k$, shall have a constant ratio to any other, since the ratio of b to k, for instance, is $b : k = r_b a : r_k a = r_b : r_k$, we have

$$\tan(l,m) = \frac{a\sin(a,m) + ar_b\sin(b,m) + ar_c\sin(c,m) + \ldots + ar_k\sin(k,m)}{a\cos(a,m) + ar_b\cos(b,m) + ar_c\cos(c,m) + \ldots + ar_k\cos(k,m)},$$

$$\text{or,}\quad \tan(l,m) = \frac{\sin(a,m) + r_b\sin(b,m) + r_c\sin(c,m) + \ldots + r_k\sin(k,m)}{\cos(a,m) + r_b\cos(b,m) + r_c\cos(c,m) + \ldots + r_k\cos(k,m)},$$

which fraction is obviously a constant quantity, being independent of a, or of the absolute length of any of the sides, $a, b, c, \ldots, k$, and depending only upon the constant quantities, $r_b, r_c, \ldots, r_k$, $\sin(a,m)$, $\sin(b,m)$, ..., $\sin(k,m)$, $\cos(a,m)$, $\cos(b,m)$, ..., $\cos(k,m)$.

The $\tan(l,m)$ being constant, the $\angle\,(l,m)$ is constant;

$\therefore$ $\angle$s, (l,a), (l,b), (l,c), ..., (l,k), are constant;

$$\text{and}\quad l = \frac{a\sin(a,m) + ar_b\sin(b,m) + \ldots + ar_k\sin(k,m)}{-\sin(l,m)}$$

$$= (\text{constant}) \cdot a. \quad \text{Q. E. D.}$$

Polygons, which, like the above, have the same number of sides proportional in the same order and their homologous angles equal, are said to be *similar*.

Cor. 2. In similar polygons, like diagonals are to each other as the homologous sides; for the diagonal is obviously in the same condition as a last side, l. (424)

Cor. 3. The perimeters, or their like portions, in similar polygons, are to each other as homologous sides or diagonals. (425)

* See Variation, Part 1, Book 1.

For $a = a,\ b = r_b a,\ c = r_c a,\ d = r_d a,\ \ldots,\ k = r_k a,\ l = r_l a$;
give $a + b + c + d + \ldots + k + l = (1 + r_b + r_c + r_d + \ldots + r_k + r_l)a.$

Lemma. Similar polygons may be described by the revolution of variable radii vectores, preserving a constant ratio to each other and the same angular motion. For let the poles be made common in O and the radii OP, Op, depart from the same [*fig*. 99.] axis of angular motion, OaA, then will OpP be a straight line for corresponding points of the polygons; and we have, since by hypothesis OP bears a constant ratio to Op and OA, Oa, OB, Ob, OC, Oc, ..., are corresponding positions of OP, Op, (426)

$$\text{OP} : \text{O}p = \text{OA} : \text{O}a = \text{OB} : \text{O}b = \text{OC} : \text{O}c = \ldots,$$

whence POA, pOa,— POB, pOb,— AOB, aOb,— BOC, bOc,— ..., are similar triangles, and AP, ap,— AB, ab,— ABC, abc,— ..., are similar parts of the perimeters of similar polygons.

Cor. 4. In similar polygons, the perimeters, or their like portions, are to each other as the corresponding radii vectores. (427)

Cor. 5. Similar curves, regarded as defined by a description altogether analogous to (426), or their like portions, are to each other as their corresponding radii vectores. For (427), depending only upon the condition that the radii vectores maintain a constant ratio, is independent of the number and magnitude of the sides of the polygons, which therefore may be made to coincide with similar curves by less than any assignable quantities. (428)

On this principle is constructed the *Pantograph*, useful for copying maps, or any kind of plane figures, whatever may be their outlines. It consists of a parallelogram of rulers, ABCp, jointed at the angles and having two of its sides, BA, BC, sufficiently produced to admit of pins being inserted in them at P and O, in a straight line with the intervening corner, p. If O be fixed, the points, P, p, will describe similar curves; so that an exact copy will be obtained, either diminished or enlarged, according as the pencil is placed at p or P—and the relative magnitudes of the figures may be varied by changing the pins, A, C. If p be fixed, P and O will describe similar figures, which may bear any ratio to each other, not excepting that of equality.

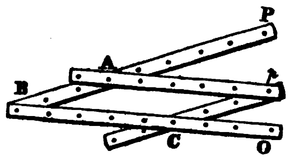


Fig. 101 2.

Scholium I. There will be no difficulty in determining (429)

the *direction* of the last side, l, if we observe that, on arriving at its first extremity, our progress will have been farther north than south, or our LATITUDE *north*, when the sum of the northings shall exceed that of the southings, or when

$$a'+b'+\ldots+k' \text{ or } a\cos(a,m)+\ldots+k\cos(k,m)>0, \text{ i. e., } +;$$

and *v. v.*; also that our position will be found to the east of the first station, or our DEPARTURE *east*, when the sum of the eastings shall exceed that of the westings, or when

$$a''+b''+\ldots+k'' \text{ or } a\sin(a,m)+\ldots+k\sin(k,m)>0, \text{ and } v.\ v.$$

Scholium II. A part of the tabular labor may be saved by making the projections upon and perpendicular to one of the sides; thus, if we assume a as a false meridian, or make $(a,m)=0$, (421) becomes (430)

$$\tan(l,a)=\frac{b\sin(b,a)+\ldots+k\sin(k,a)}{a+b\cos(b,a)+\ldots+k\cos(k,a)}$$

and (422), $$l=\frac{b\sin(b,a)+\ldots+k\sin(k,a)}{-\sin(l,a)}.$$

Scholium III. By eliminating between (420) and (419) any two sides, as k and l, may be determined when all the other quantities are given; but it will be better to operate as if l were a meridian, when (l,m) will $=0$, $(a,m)=(a,l)$, $(b,m)=(b,l)$, $(c,m)=(c,l)$, and from (420), (419), there will result (431)

$$k=\frac{a\sin(a,l)+b\sin(b,l)+c\sin(c,l)+\ldots+j\sin(j,l)}{-\sin(k,l)},$$

$$-l=a\cos(a,l)+b\cos(b,l)+c\cos(c,l)+\ldots+k\cos(k,l).$$

EXERCISES.

1°. Required the last side in I.

Operation.

$$\begin{array}{ccc}[E,+] & [W,-] & [W,-].\end{array}$$

$$\frac{15{\cdot}75\sin 36{\cdot}5^\circ-20\sin 15{\cdot}3^\circ-30{\cdot}25\sin 46^\circ}{15{\cdot}75\cos 36{\cdot}5^\circ+20\cos 15{\cdot}3^\circ-30{\cdot}25\cos 46^\circ}=\tan(d,m).$$

$$\begin{array}{ccc}[N,+] & [N,+] & [S,-].\end{array}$$

logs. nos	1·19728		1·30103		1·48073
logs. sins	$\bar{1}$·77439		$\bar{1}$·42140		$\bar{1}$,85693
logs. coss	$\bar{1}$·90518		$\bar{1}$·98433		$\bar{1}$·84177
log. a'' =	0·97167	log. b'' =	0·72243	log. c'' =	1·33766
log. a' =	1·10246	log. b' =	1·28536	log. c' =	1·32250

$a'' = a \sin(a,m) = +\ 9·3685$ $\qquad$ $a' = a \cos(a,m) = +\ 12·661$

$b'' = b \sin(b,m) = -\ 5·2775$ $\qquad$ $b' = b \cos(b,m) = +\ 19·291$

$c'' = c \sin(c,m) = -\ 21·7600$ $\qquad$ $c' = c \cos(c,m) = -\ 21·014$

numerator $= -\ 17·6690$ $\qquad$ denominator $= +\ 10·938$

$\therefore$ eastings < westings. $\qquad$ $\therefore$ northings > southings.

log. 17·669 = 1·24721	log. 17·669 = 1·24721
log. 10·938 = 1·03894	log. sin 58·24° = $\bar{1}$·92955
log. tan (d,m) = 0·20827	log. d = 1·31766
$\therefore$ (d,m) is S. 58·24° E.	$\therefore$ d = 20·781

If, in accordance with the second scholium, we assume a for the meridian, whose bearing is N 36·5° E, the NW and SE bearings will be increased by 36·5°, and the NE and SW diminished by the same angle; hence Field Notes, No. I., will be transformed into

a	N_1 00·0° E_2	15·75 chs.
b	N_2 51·8° W_1	20·00
c	S_2 9·5° W_1	30·25
d		

The student will calculate d according to this table.

2°. Calculate e of II by (421) and (422).

3°. Calculate f of III twice, by projecting first on b then on c, and balance the errors by adding together and dividing by 2.

4°. Calculate g of IV and verify by a different method.

5°. Find AG in V.

6°. Calculate the diagonal drawn from the first extremity of a in IV to the second extremity of c.

7°. Solve VI, finding also the length, l, of the intercepted meridian. [l intersects e : why ?]

$$k = \frac{-11{\cdot}5\sin 69^\circ - 17\sin 28{\cdot}4^\circ + 11\sin 31{\cdot}6^\circ + 15\sin 56{\cdot}8^\circ}{-\sin 67{\cdot}1^\circ} = 0{\cdot}55,$$

$$l = 11{\cdot}5\cos 69^\circ + 17\cos 28{\cdot}4^\circ + 11\cos 31{\cdot}6^\circ + 15\cos 56{\cdot}8^\circ - 0{\cdot}55\cos 67{\cdot}1^\circ = 36{\cdot}444.$$

8°. Required the length, l, and second point of intersection with the perimeter, of a line running from $\angle\,(a,b)$ of IV, N 25° W.

PROPOSITION II.

It is required to find the area of a polygon in terms of the sides and the angles which these sides make with each other.

The easiest way of ascertaining the area of any polygon $(a, b, c, \dots, j, k, l)$, is obviously to divide it into triangles (A, b), (A, c), (A, d), ..., (A, j), (A, k), and then to compute these triangles. But the double area of a triangle is had at once by taking the product of its base and altitude. Therefore from A, the common vertex of all the triangles, and upon their bases, $b, c, d, \dots, j, k$, produced, if necessary, let fall the corresponding perpendiculars, $p_b, p_c, p_d, \dots, p_j, p_k$; there results

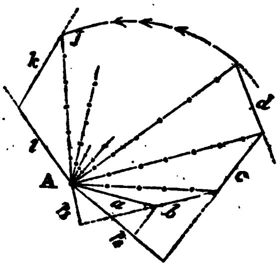


Fig. 102.

$$\begin{aligned} 2\text{triangle }(A,b) &= bp_b,\\ 2\text{tr}(A,c) &= cp_c,\\ 2\text{tr}(A,d) &= dp_d,\\ \&c., \quad & \quad \&c.,\\ 2\text{tr}(A,j) &= jp_j,\\ 2\text{tr}(A,k) &= kp_k. \end{aligned}$$

But p_b is obviously the projection of a upon the line of p_b, p_c the sum of the projections of a and b upon the line of p_c, p_d the sum of the projections of a, b, c, upon the line of p_d, ..., p_j the sum of the projections of $a, b, c, \dots i$, upon the line of p_j, p_k the sum of the projections of $a, b, c, \dots, k$, upon the line of p_k;

$$\therefore\ p_b = a\cos(a,p_b) = a\sin(a,b), \text{ since } \angle\,(a,p_b) + (a,b) = 90^\circ;$$

so
$$\begin{aligned} p_c &= a\sin(a,c) + b\sin(b,c),\\ p_d &= a\sin(a,d) + b\sin(b,d) + c\sin(c,d),\\ &\&c., \quad \&c., \quad \&c., \quad \&c.,\\ p_j &= a\sin(a,j) + b\sin(b,j) + c\sin(c,j) + \dots + i\sin(i,j),\\ p_k &= a\sin(a,k) + b\sin(b,k) + c\sin(c,k) + \dots + j\sin(j,k); \end{aligned}$$

therefore, substituting above, and adding, there results,

$$2P=\begin{cases} 2(A,b) \\ +2(A,c) \\ +2(A,d) \\ +\ \&c. \\ +2(A,k) \end{cases}=\begin{cases} ab\sin(a,b) \\ +ac\sin(a,c)+bc\sin(b,c) \\ +ad\sin(a,d)+bd\sin(b,d)+cd\sin(c,d) \\ +\&c.,\quad \&c.,\quad \&c., \\ +ak\sin(a,k)+bk\sin(b,k)+\ldots+jk\sin(j,k); \end{cases} \quad (432)$$

∴ *The double area of any polygon is equal to the sum of the products of its sides, save one, multiplied, two and two, into the sines of the angles formed by the sides belonging to the several products.*

Cor. 1. The double area of a triangle is equal to the product of two of its sides multiplied into the sine of the angle included by them. $2\text{tr}(a,b,c) = ab\sin(a,b)$. (433)

Cor. 2. The area of a parallelogram is equal to the product of its dimensions multiplied into the sine of the included angle (433). (434)

Cor. 3. When two triangles have an angle of the one supplementary to an angle of the other, the triangles are to each other as the products of the sides about the supplementary angles (433). (435)

Cor. 4. Combining (433) and (393),

or $$2\text{tr}(a,b,c) = ab\sin(a,b)$$

and $$\sin(a,b) = \frac{2[h(h-a)(h-b)(h-c)]^{\frac{1}{2}}}{ab};$$

there results

$$\text{tr}(a,b,c) = [h(h-a)(h-b)(h-c)]^{\frac{1}{2}}; \text{ i. e.} \quad (436)$$

The area of a triangle is equal to the square root of the continued product of the half sum of the three sides and the three remainders formed by diminishing this half sum by the sides severally. This furnishes a rule very convenient for the application of logarithms.

Cor. 5. Similar polygons and their like segments and sectors are to each other as the squares of their homologous lines, whether sides, diagonals, or radii vectores. (437)

For, preserving the notation under (423) and substituting in (432) there results

$$2P = \left\{\begin{array}{l} r_b \sin(a,b) \\ + r_c \sin(a,c) + r_b r_c \sin(b,c) \\ + r_d \sin(a,d) + r_b r_d \sin(b,d) + r_c r_d \sin(c,d) \\ + \text{ \&c., \&c., \&c.,} \\ + r_k \sin(a,k) + r_b r_k \sin(b,k) + \ldots + r_j r_k \sin(j,k) \end{array}\right\} \cdot a^2. \qquad (438)$$

Cor. 6. If the polygon is equilateral, that is

$$a = b = c = \ldots = j = k = l,$$

or

$$r_b = r_c = r_d = \ldots = r_j = r_k = 1,$$

we have

$$2P = \left\{\begin{array}{l} \sin(a,b) \\ + \sin(a,c) + \sin(b,c) \\ + \sin(a,d) + \sin(b,d) + \sin(c,d) \\ + \text{ \&c., \&c., \&c.,} \\ + \sin(a,k) + \sin(b,k) = \ldots + \sin(j,k) \end{array}\right\} \cdot a^2. \qquad (439)$$

Cor. 7. If the polygon be regular, that is, have its angles as well as its sides equal, putting

$$\angle\,(a,b) = e,$$

and therefore, $(a,c) = 2e$, $(b,c) = e$,

$$(a,d) = 3e,\ (b,d) = 2e,\ (c,d) = e,$$
$$\text{\&c.,} \quad \text{\&c.,} \quad \text{\&c.,}$$
$$(a,k) = (n-1)e,\ (b,k) = (n-2)e,$$
$$(c,k) = (n-3)e,\ \ldots,\ (j,k) = e,$$

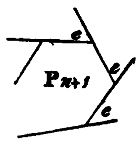


Fig. 102a.

$n + 1$ denoting the number of sides of the polygon, there results, (439),

$$2P_{n+1} = \left\{\begin{array}{l} \sin e \\ + \sin 2e + \sin e \\ + \sin 3e + \sin 2e + \sin e \\ + \text{ \&c., \&c., \&c.,} \\ + \sin(n-1)e + \sin(n-2)e + \sin(n-3)e + \ldots + \sin e \end{array}\right\} \cdot a^2,$$

or
$$\left.\begin{array}{l} 2P_{n+1} = [(n-1)\sin e + (n-2)\sin 2e \\ \quad + (n-3)\sin 3e + \ldots + \sin(n-1)e] \cdot a^2. \end{array}\right\} \qquad (440)$$

Cor. 8. Making $n = 2$, we find

$$P_3 = \tfrac{1}{2}\left[(2-1)\sin\frac{360^\circ}{3}\right] \cdot a^2 = \tfrac{1}{2}\sin 60^\circ \cdot a^2 = 3^{\frac{1}{2}}(\tfrac{1}{2}a)^2, \qquad (441)$$

for the area of the *Equilateral Triangle.*

Cor. 9. Making $n = 3$, we find

$$P_4 = \tfrac{1}{2}[2\sin 90^\circ + \sin 180^\circ] \cdot a^2 = a^2, \qquad (442)$$

for the area of the *Square*, as already known.

Cor. 10. Making $n = 4$, we find

$$P_5 = \tfrac{1}{2}\sin 36^\circ(1 + 6\cos 36^\circ) \cdot a^2$$

$$= \frac{(5+3\cdot 5^{\frac{1}{2}})\,(2\cdot 5 - 2\cdot 5^{\frac{1}{2}})^{\frac{1}{2}}}{16} \cdot a^2, \qquad (443)$$

for the area of the *Regular Pentagon.*

Cor. 11. For $n = 5$, or the *Regular Hexagon,*

$$P_6 = a^2 \cdot 3\sin 60^\circ = a^2 \cdot \frac{3}{2}\sqrt{3}. \qquad (444)$$

Cor. 12. Similar plane figures, whether bounded by (445) straight lines or curves, are to each other as the squares of their homologous arcs, chords and radii vectores, (437,) (426).

Scholium I. It is obvious, from the course of the de- (446) monstration of (432), that the angles must be all estimated in the same direction, either from the right round to the left, or from the left to the right. And the rules for the algebraical signs of the trigonometrical lines, as everywhere else, are here also to be rigorously observed.

Thus, $\sin 211^\circ = \sin(180^\circ + 31^\circ) = -\sin 31^\circ.$

For the computation of the area of the first field, we have

$$2P = ab\sin(a,b) \qquad [2(A,b)]$$
$$+ ac\sin(a,c) + bc\sin(b,c); \quad [2(A,c)]$$

$$\therefore \quad \angle(a,b) = 36{\cdot}5^\circ + 15{\cdot}3^\circ = 51{\cdot}8^\circ,$$
$$(a,c) = 180^\circ + 36{\cdot}5^\circ - 46^\circ = 170{\cdot}5^\circ,$$
$$(b,c) = 180^\circ - (15{\cdot}3^\circ + 46^\circ) = 118{\cdot}7^\circ.$$

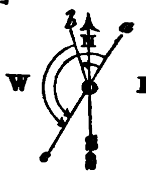


Fig. 102_2.

Operation.

a, b	51·8°	$\bar{1}$·89534	2·39365	
a, c	170·5°	$\bar{1}$·21761	1·89562	
b, c	118·7°	$\bar{1}$·94307	2·72383	
a	15·75	1·19728	247·55	[2(A,b)]
b	20·00	1·30103	78·64	[2(A,c)]
c	30·25	1·48073	529·45	

$$\therefore \quad 2P = 855{\cdot}64$$

$$\therefore \quad P = 427{,}82 \text{ sq. chs.}$$
$$= 42{\cdot}782 \text{ acres.}$$

Operation for P *of* II.

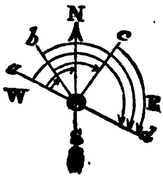


Fig. 102$_4$.

180°

a N 80° W
b N 2° W
c N 86° E
d S 32° E

a, *b*	78°	$\bar{1}$·990 4044	3·463 1495							290·502	290·502=2(*A*,*b*)
a, *c*	166°	$\bar{1}$·383 6752	2·804 3513							63·731	
b, *c*	88°	$\bar{1}$·999 7354	3·576 5612							377·191	440·922=2(*A*,*c*)
a, *d*	228°	$\bar{1}$·871 0735	3·228 3412							− 169·177	
b, *d*	150°	$\bar{1}$·698 9700	3·212 3874							163·075	
c, *d*	62°	$\bar{1}$·945 9349	3·407 2833							255·437	249·335=2(*A*,*d*)
a	45·53	1·658 2977	+	+		+				980·759	
b	65·23	1·814 4474	+		+		+			490·379 acres = *P*	
c	57·86	1·762 3784		+	+			+			
d	50·00	1·698 9700				+	+	+			

Scholium II. The operation may be materially shortened (447) by making a diagonal the last or excepted side, and computing the polygon in two distinct parts; also gross errors will be detected, and the slight ones, always incident to mathematical tables and instrumental admeasurements, corrected, by changing the diagonal.

Thus, if we imagine a diagonal joining the extremities of *a* and *c* in II, we shall find

$$(a,b,c) = 365{\cdot}711 \text{ acres},$$
$$(d,e) = 124{\cdot}669 \text{ do.};$$
$$\therefore \quad (a,b,c,d,e) = 490{\cdot}380.$$

Again, let the diagonal join the extremities of *b* and *d*, and there results,

$$(b,c,d) = 397{\cdot}852,$$
$$(e,a) = 92{\cdot}529,$$
$$\therefore \quad (a,b,c,d,e) = 490{\cdot}381;$$
so
$$(c,d,e) = 345{\cdot}131,$$
$$(a,b) = 145{\cdot}251,$$
$$\therefore \quad (a,b,c,d,e) = 490{\cdot}382;$$

$$\therefore \quad 3(a,b,c,d,e) = 1471{\cdot}143,$$
and
$$(a,b,c,d,e) = 490{\cdot}381.$$

Scholium III. When the sides of the polygon are numerous, it will be expedient to divide it into parts by one or more diagonals determined by computation or by actual measurement. (448)

EXERCISES.

1°. Calculate the area (a,b,c,d) of I by calculating the parts (a,b), (c,d).

2°. The same by computing $(b,c)+(d,a)$.

3°. Balance errors by taking the half sum of 1° and 2°.

4°. Find the area of I by excepting d.

5°. The same by excepting a.

6°. The same by excepting b.

7°. The same by excepting c.

8°. Balance errors.

9°. The area of II may be calculated by several of the following methods:

$(a,b,c)+(d,e)$, $(b,c,d)+(e,a)$, $(c,d,e)+(a,b)$, $(d,e,a)+(b,c)$, $(e,a,b)+(c,d)$;

(a,b,c,d), (b,c,d,e), (c,d,e,a), (d,e,a,b), (e,a,b,c).

10°. Employ some of the fifteen methods following for the computation of the area of III.

$(a,b,c)+(d,e,f)$, $(b,c,d)+(e,f,a)$, $(c,d,e)+(f,a,b)$;

$(a,b,c,d)+(e,f)$, $(b,c,d,e)+(f,a)$, $(c,d,e,f)+(a,b)$, $(d,e,f,a)+(b,c)$, $(e,f,a,b)+(c,d)$, $(f,a,b,c)+(d,e)$;

(a,b,c,d,e), (b,c,d,e,f), (c,d,e,f,a), (d,e,f,a,b), (e,f,a,b,c), (f,a,b,c,d).

11°. Write out all the methods for the computation of the area of IV, and execute a number of them.

12°. Calculate the area lying on the west of the meridian which passes through the middle point of the side a of IV.

Ans. 44·812 acres.

13°. Compute, by the aid of logarithms and (436), the area of a triangle the sides of which are 13·33⅓, 15·75, 16·02¼ chains.

14°. Find the area of the pentagon given under "*the Chain.*"

PROPOSITION III.

To cut off from a polygon a given area, PBCDX, *by* (449) *a line,* PX, *running from a given point,* P, *in one of the sides, as* AB.

Compute the triangles PBC, PCD, PDE, ..., till two consecutive areas, PBCD, PBCDE, be found, the first less, the second greater than the required area, PBCDX; then will it be known on what side, DE, the point X must fall. The areas DPE, DPX, being known, the one by computation and the other by hypothesis, we have the proportion

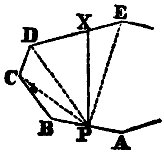


Fig. 103.

$$\frac{\mathrm{DPE}}{\mathrm{DPX}} = \frac{\mathrm{DE}}{\mathrm{DX}}$$

for calculating DX, when XP will be determined by the process for a last side.

As an example, let it be required to cut off 200 acres from the left of II, by a line running from the first station A. By referring to the computation above, we find the point X will fall on c, and the operation may be put down as follows:

$\frac{\mathrm{tr}(A,c)}{\mathrm{tr}(A,x)}$	2204·61 sq. chs.	3·343332
	547·49 sq. chs.	2·738376
$= \frac{c}{x}$	57·86 chs.	1·762378
	$\therefore x = 14{\cdot}369$ chs.	1·157422

PROPOSITION IV.

To lay out a given area in a triangle of a given form.

Let A, B, C, denote the angles severally opposite the sides a, b, c, ; then (432), (337),

$$2\mathrm{P} = ab \sin(a,b) = ab \sin\mathrm{C},$$
$$2\mathrm{P} = bc \sin(b,c) = bc \sin\mathrm{A},$$
$$2\mathrm{P} = ca \sin(c,a) = ca \sin\mathrm{B};$$

$$\therefore \quad 2\mathrm{P} = \frac{\sin\mathrm{B}\sin\mathrm{C}}{\sin\mathrm{A}} \cdot a^2. \tag{450}$$

Cor. 1. If the triangle be isosceles, or C = B, whence

$$180^\circ = A + B + C = A + 2B,$$
$$B = C = 90^\circ - \tfrac{1}{2}A,$$
$$\sin B = \sin C = \sin(90^\circ - \tfrac{1}{2}A) = \cos\tfrac{1}{2}A,$$
$$\frac{\sin B \sin C}{\sin A} = \frac{\cos\frac{1}{2}A \cos\frac{1}{2}A}{\sin A} = \frac{\cos\frac{1}{2}A \cos\frac{1}{2}A}{2\sin\frac{1}{2}A \cos\frac{1}{2}A} = \tfrac{1}{2}\cot\tfrac{1}{2}A,$$

there results

$$P = \frac{a^2}{4}\cot\tfrac{1}{2}A. \tag{451}$$

Cor. 2. If the isosceles triangle P be applied m times about its vertex A, to form* the regular polygon, P_m, we have,

$$P_m = mP = \frac{ma^2}{4}\cot\tfrac{1}{2}A,$$

or

$$P_m = \frac{ma^2}{4}\cot\frac{180^\circ}{m}, \tag{452}$$

since $mA = 360^\circ$.

What will (452) become for $m = 3, 4, 5, 6, 7, 8, \ldots$?

For illustration, let it be required to lay out ten acres in a triangle whose angles are, $A = 45^\circ$, $B = 87^\circ$.

We have (450)

$$a = \left[\frac{200 \sin 45^\circ}{\sin 87^\circ \sin 48^\circ}\right]^{\frac{1}{2}};$$

$$\therefore \quad a = 13{\cdot}80 \text{ chs.}$$

For the side of an octagon of 15 acres, we find

$$150 = \frac{8}{4}a^2 \cot\frac{180^\circ}{8},$$

or

$$75 = a^2 \cot 22\tfrac{1}{2}^\circ;$$

$$\therefore \quad a = \left[\frac{75}{\cot 22\frac{1}{2}^\circ}\right]^{\frac{1}{2}} = [75 \tan 22\tfrac{1}{2}^\circ]^{\frac{1}{2}} = 31{\cdot}07 \text{ chs.}$$

PROPOSITION V.

To lay out a trapezoid of a given area, P, *on a base,* a, *also given, knowing the inclinations of the oblique sides,* x, y, *to* a.

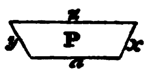


Fig. 104.

We have

$$xa\sin(x,a) + xy\sin(x,y) + ay\sin(a,y) = 2P,$$
$$ay\sin(a,y) + az\sin(a,z) + yz\sin(y,z) = 2P,$$
$$zx\sin(z,x) + za\sin(z,a) + xa\sin(x,a) = 2P;$$

---

* The student will construct the figure.

but $\sin(y,z) = \sin(a,y)$, $\sin(z,x) = \sin(x,a)$, $\sin(a,z) = \sin(z,a) = 0$;
$\therefore$ the second and third equations reduce to

$$y(a+z)\sin(a,y) = 2P,$$
$$x(a+z)\sin(x,a) = 2P; \qquad (453)$$

$$\therefore \qquad y = \frac{\sin(x,a)}{\sin(a,y)} \cdot x; \qquad (454)$$

by which the first equation reduces to

$$x^2 + 2\left[a \cdot \frac{\sin(a,y)}{\sin(x,y)}\right] \cdot x = \frac{2P}{\sin(x,a)} \cdot \frac{\sin(a,y)}{\sin(x,y)}. \qquad (455)$$

The coëfficients $a \cdot \dfrac{\sin(a,y)}{\sin(x,y)}$, $\dfrac{2P}{\sin(x,a)} \cdot \dfrac{\sin(a,y)}{\sin(x,y)}$, being calculated, this equation may be reduced as an ordinary quadratic, but better by (385_2), observing that

$$x + p = \pm \frac{p}{\cos v}, \quad \tan v = \frac{q^{\frac{1}{2}}}{p}, \quad p = a \cdot \frac{\sin(a,y)}{\sin(x,y)}, \qquad (456)$$

$$q = \frac{2P}{\sin(x,a)} \cdot \frac{\sin(a,y)}{\sin(x,y)}; \qquad (456)$$

and x thus becoming known (454) determines y, when (453) gives $(a+z)$ and, consequently, z.

Let it be required to cut off 15 acres by a line parallel to b of I. Making $b = a = 20$, the operation will be as follows,

$$\left.\begin{array}{l} x \text{ N } 36{\cdot}5^\circ \text{ E} \\ a \text{ N } 15{\cdot}3^\circ \text{ W} \\ y \text{ S } 46^\circ \text{ W} \end{array}\right\} \therefore \left\{\begin{array}{l} (a,y) = 118{\cdot}7^\circ \\ (x,y) = 170{\cdot}5^\circ \\ (x,a) = 51{\cdot}8^\circ \end{array}\right.$$

$$\begin{aligned}
\log.\sin(a,y) &= \bar{1}{\cdot}943072 \\
\log.\sin(x,y) &= \bar{1}{\cdot}217609 \\
\log.\sin(a,y) - \log.\sin(x,y) &= 0{\cdot}725463 \\
\log. 2P &= 2{\cdot}477121 \\
\log.\sin(x,a) &= \bar{1}{\cdot}895343 \\
\log. 2P - \log.\sin(x,a) &= 2{\cdot}581778 \\
\log. q &= 3{\cdot}307241 \\
\log. q^{\frac{1}{2}} &= 1{\cdot}653620
\end{aligned}$$

$$\log. a = 1{\cdot}301030$$

$$\log. p = 2{\cdot}026493$$

$$\log. \tan v = \log. q\tfrac{1}{2} - \log. p = \bar{1},627127$$

$$\log. \cos v = \bar{1}{\cdot}964237$$

$$\log.(x+p) = \log. p - \log. \cos v = 2{\cdot}062256$$

$$x + 106{\cdot}3 = \pm 115{\cdot}4$$

$$x = 9{\cdot}1$$

$$y = \frac{\sin(x,a)}{\sin(a,y)} \cdot x = 8{\cdot}15$$

$$a + z = \frac{2P}{x \cdot \sin(x,a)} = 41{\cdot}95$$

$$\therefore \quad z = 21{\cdot}95$$

It may sometimes be desirable to execute the above problem without the aid of tables, or any other angular instrument than the cross. For this purpose measure in any convenient position a line, p, perpendicular to a, and through its extremity a parallel, b, determining the trapezoid (a,b), which should differ as little as one may judge from (a,c), the area to be laid out.

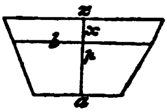


Fig. 104$_2$.

The area of the trapezoid (b,z) thus becoming known, it remains only to find its breadth, x; and for this purpose we have

$$\text{trap} \cdot (b,z) = \frac{x(b+z)}{2}, \text{ and } \frac{z-b}{b-a} = \frac{x}{p};$$

$$\therefore \quad x^2 + \frac{2bp}{b-a} \cdot x = \frac{2p \text{ trap}(b,z)}{b-a}. \qquad (457)$$

PROPOSITION VI.

To cut off a given area, XABCDEFY, *from a given field, by a line,* YX, *running in a given direction.*

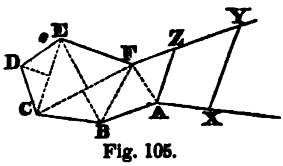


Fig. 105.

Having ascertained what sides YX will intersect,* find AZ parallel to XY,

* This may be done, if preferred, by dividing the plotted field ABCDEF into triangles, taking the measures of their bases and perpendiculars from a scale of equal parts, and finding the sum of their areas; which, however, must not be employed any farther; for when FZ, ZA, have been found by calculation, or by actual measurement in the field, the area of ABCDEFZ must be accurately computed.

and compute the area ABCDEFZ; when the area of the trapezoid XAZY becoming known its parts will be found by Proposition V.

PROPOSITION VII.

Given the length and direction of the line AB, *intersecting the lines* AK, BL, *also given in direction; to draw a line*, KL, *in a given direction and intersecting* AB *in* I, *so as to make, on opposite sides of* AB, *the areas* IAK, IBL, *equal.*

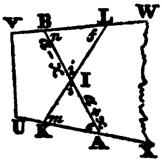


Fig. 106.

Representing the lines IA, IB, by $a+x$, $a-x$, and the angles IAK, ILB, IKA, IBL, by e, f, m, n, we have (450)

$$\frac{(a+x)^2 \sin e \sin i}{\sin m} = 2\mathrm{tr}(\mathrm{IAK}) = 2\mathrm{tr}(\mathrm{IBL}) = \frac{(a-x)^2 \sin n \sin i}{\sin f};$$

$$\therefore (a+x)\left(\frac{\sin e}{\sin m}\right)^{\frac{1}{2}} = (a-x)\left(\frac{\sin n}{\sin f}\right)^{\frac{1}{2}};$$

$$\therefore x = \frac{\left(\frac{\sin n}{\sin f}\right)^{\frac{1}{2}} - \left(\frac{\sin e}{\sin m}\right)^{\frac{1}{2}}}{\left(\frac{\sin n}{\sin f}\right)^{\frac{1}{2}} + \left(\frac{\sin e}{\sin m}\right)^{\frac{1}{2}}} \cdot a = \frac{1 - \left(\frac{\sin e \sin f}{\sin m \sin n}\right)^{\frac{1}{2}}}{1 + \left(\frac{\sin e \sin f}{\sin m \sin n}\right)^{\frac{1}{2}}} \cdot a;$$

$$\left.\begin{array}{ll} \text{or (354), } & x = \dfrac{1-\tan v}{1+\tan v} \cdot a = a\tan(45^\circ - v), \\ \text{putting} & \tan v = \left(\dfrac{\sin e \sin f}{\sin m \sin n}\right)^{\frac{1}{2}}. \end{array}\right\} (458)$$

Scholium I. We observe that (458), being well adapted to logarithmic computations, may be advantageously combined with (449) for the solution of Propositions V and VI, also in changing the direction of a line between two farms.

Thus, let it be required to cut off a given area, KUVL, by a line, LK, running in a given direction. In the line VL take any point, B, and by (449) determine the point A in the line of UK, such that AUVB may = the required area, KUVL, and apply (458). It will obviously facilitate the operation if it be convenient to reduce B to coincidence with V. The same process will change AB, the division line of two farms, into a second required position, KL.

Scholium II. The intelligent student will not find any difficulty in applying the propositions now given to the straightening of a broken boundary between estates.

PROPOSITION VIII.

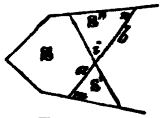


Fig. 167.

Through a point, P, *given in position in a given polygon, to run a line cutting off a required area,* $S + S'$.

By calculation or actual measurement, determine any convenient line passing through the given point and separated by it into the parts a, b; also compute the included area $S + S''$; it follows that, since $S + S'$ and $S + S''$ are given quantities, their difference, $S' - S''$ becomes known. But we have

$$\frac{a^2 \sin m \sin i}{\sin(m+i)} = 2S' \text{ and } \frac{b^2 \sin n \sin i}{\sin(n+i)} = 2S''.$$

$$\therefore \quad \frac{a^2 \sin m \sin i}{\sin(m+i)} - \frac{b^2 \sin n \sin i}{\sin(n+i)} = 2(S' - S''), \text{ which put} = c,$$

or
$$\frac{a^2 \sin m \sin i}{\sin m \cos i + \cos m \sin i} - \frac{b^2 \sin n \sin i}{\sin n \cos i + \cos n \sin i} = c,$$

or
$$\frac{a^2 \sin m}{\sin m \cot i + \cos m} - \frac{b^2 \sin n}{\sin n \cot i + \cos n} = c;$$

$$\begin{aligned} \therefore \quad & a^2 \sin m \sin n \cot i + a^2 \sin m \cos n \\ & - b^2 \sin m \sin n \cot i - b^2 \cos m \sin n \\ & = c \sin m \sin n \cot^2 i + c(\sin m \cos n + \cos m \sin n)\cot i \\ & + c \cos m \cos n; \end{aligned}$$

$$\therefore \quad \cot^2 i + \left[\frac{\sin(m+n)}{\sin m \sin n} + \frac{b^2 - a^2}{c}\right]\cot i$$
$$= \frac{a^2 \cot n - b^2 \cot m}{c} - \cot m \cot n,$$

or, $$\cot^2 i + \left[\cot n + \frac{(b+a)(b-a)}{c}\right]\cot i = \frac{a^2 \cot n}{c},$$

if it be convenient to make $m = 90°$. (459)

PART THIRD.

SOLID GEOMETRY, SPHERICAL GEOMETRY, AND NAVIGATION.

BOOK FIRST.

SOLID GEOMETRY.

SECTION FIRST.

Planes.

PROPOSITION I.

Three points not in the same straight line determine the position of a plane. (460)

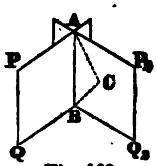


Fig. 108.

For, let the plane, P, pass through two of the points, as A, B; then, revolving upon these points, it will become fixed or determined in position, when it shall contain the third point, C.

Cor. 1. Two intersecting lines, as AC, BC, determine the position of a plane. (461)

Cor. 2. A triangle, as ABC, is always in the same plane and determines its position. (462)

Cor. 3. A plane is determined in position by two parallel lines, as AP, BQ. (463)

Cor. 4. Two planes cannot intersect each other in more lines than one. (464)

For if A and B be any points common to the planes P and P_2, it is obvious from the definition of a plane that the straight line AB will lie wholly in both planes, and will therefore be a line of intersection. Now the planes, P, P_2, cannot have a second line of intersection, as ACB; since this hypothesis would reduce the planes to coincidence (460), the three points, A, C, B, not in the same straight line, becoming common to P and P_2.

Cor. 5. The intersections of planes are straight lines. (465)

Cor. 6. Planes coinciding in three points, not in the same straight line, coincide throughout. (466)

PROPOSITION II.

A line drawn through the intersection of two other lines and perpendicular to both of them, will be perpendicular to their plane. (467)

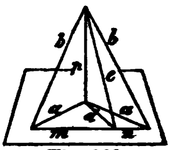


Fig. 109.

For let p be the perpendicular, a, a, equal portions of the intersecting lines, b, b, the hypothenuses to p, a, $-p$, a, which will therefore be equal, and let d be any line drawn through the angle (a, a) and terminating in the line $m+n$ joining the extremities of a, a, and divided by d into the parts m, n; it only remains to show that e, joining the extremities of p, d, is a hypothenuse. We have (137)

$$a^2 - d^2 = mn = b^2 - e^2;$$

$$\therefore \quad e^2 - d^2 = b^2 - a^2 = p^2, \text{ Q. E. D., (133).}$$

Cor. 1. If one side of a right angle be made a fixed axis of revolution, the other side, in revolving, will describe a plane. (468)

For, if the right angle (a, p) revolve about p as an axis, a will be found constantly in the plane of (a, a).

Cor. 2. Of oblique lines drawn from any point in a perpendicular to a plane and terminating in this plane, the more distant will be the greater, those equally distant will be equal and terminate in the circumference of the same circle having the foot of the perpendicular for centre. (469)

Cor. 3. Planes which are perpendicular to the same straight line are parallel to each other; and, conversely, if a line be perpendicular to one of two parallel planes, it will be perpendicular to the other also. (470)

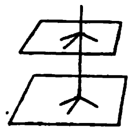


Fig. 109_2.

Cor. 4. If from any point without a plane, a perpendicular be dropped upon the plane, and from the foot of this perpendicular a second perpendicular be let fall upon any line in the plane the line joining the first and last-mentioned points, will be perpendicular to the line drawn in the plane. (471)

For if $m=n$, d and e will both be perpendicular to $m+n$.

Cor. 5. Through the same point, either without or within a plane, but a single perpendicular can be drawn. (472)

For let e be any line intersecting the perpendicular, p; e is obviously inclined to d and therefore to the plane (a, a).

Cor. 6. A plane passing through a perpendicular to a second plane, is perpendicular to the same. (473)

For let d revolve to take up a position perpendicular to a; then (p, d) being a right angle, the plane (p, a) is said to be perpendicular to the plane (a, a, d).

Cor. 7. The intersection of two planes perpendicular to a third, is perpendicular to the same plane. (474)

Cor. 8. Lines perpendicular to the same plane are parallel to each other. (475)

For let p, p_2, p_3, ..., be lines perpendicular to the plane, P, and a, b, ..., the lines joining the points in which the perpendiculars intersect the plane; it follows from (473), (474), that the planes (p, a), (p_2, b), will intersect in p_2; whence p, p_2, being perpendicular to a, p_2, p_3, to b, ..., p, p_2, p_3, are parallel to each other.

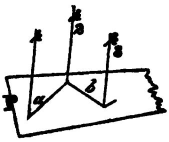


Fig. 109₂.

Cor. 9. A line parallel to a perpendicular to a plane is itself a perpendicular to the same plane. (476)

Cor. 10. Lines parallel to the same line situated any way in space, are parallel to each other. (477)

For they will be perpendicular to the same plane.

PROPOSITION III.

If a plane cut parallel planes, the lines of intersection will be parallel. (478)

For, if the intersections m, n, of the plane, P, with the parallel planes, M, N, were not parallel, but met on being produced, then would M, N, cut each other in the same point, which is contrary to the hypothesis.

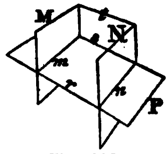


Fig. 110.

Cor. 1. The segments, as r, s, t, of parallel lines intercepted by parallel planes, are equal. (479)

Cor. 2. Conversely, two planes intercepting equal seg- (480)

ments of three parallel lines not situated in the same plane, are parallel.

Cor. 3. Parallel planes are everywhere equally distant. (481) [Let r, s, t, be perpendicular to M, N.]

Cor. 4. Two angles, having their sides parallel and opening in the same direction, are equal, and their planes are parallel. (482)

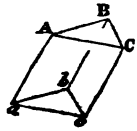


Fig. 110₂.

For, let the sides AB, AC, of the angle A, be parallel respectively to the sides ab, ac, of the angle a, and open in the same direction; draw Bb, Cc, parallel to Aa, then will the quadrilaterals, Ab, Ac, Bc, be parallelograms, and the sides of the triangles BAC, bac, severally equal,— $\therefore$ $\angle A = a$; but Aa = Bb = Cc, $\therefore$ (480) the plane BAC will be parallel to the plane bac.

Cor. 5. A *Dihedral* angle, or the angle which one plane makes with another, is measured by the inclination of two perpendiculars drawn through the same point in its edge, one in each side. (483)

For, make Aa perpendicular to the planes BAC, bac, then will Aa be perpendicular to AB, AC, ab, ac, and the dihedral angle BAac will be measured by the plane angle BAC = bac; from which it follows that the point A, through which the perpendiculars AB, AC, are drawn, may be taken anywhere in the edge, Aa, of the dihedral angle.

Cor. 6. The segments of lines intercepted by parallel planes are proportional. (484)

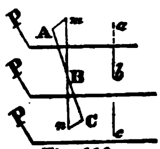


Fig. 110₃.

For, let ABC, abc, be any lines whatever, piercing the parallel planes, P, P, P, in the points A, B, C, a, b, c; and through B draw mBn parallel to abc, piercing the planes in m, n. Since A, B, C, m, n, are in the same plane, and mA parallel to Cn, we have

$$AB : BC = mB : Bn = ab : bc.$$

PROPOSITION IV.

If a line passing through a fixed point revolve in any manner so as constantly to intersect two parallel planes, the figures thus described will be similar. (485)

For, let VaA, VxX, VyY, ..., be positions of the revolving line passing through the fixed point, V, and piercing the parallel planes in A, a, X, x, Y, y, ... ; drop the perpendicular VpP, piercing the planes in P, p, and join PA, PX, pa, px. The radii vectores PX, px, have the constant ratio $Vp : Vp$, and $\angle APX = apx$; $\therefore$ (426) the plane figures AXY, ..., axy, ..., are similar.

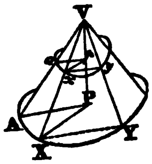


Fig. 111.

Definition. The solid (V, AXY ...) is denominated a *Cone*, when the perimeter AXY ... is wholly curvilinear, and it becomes a *Pyramid* when AXY ... is made up of straight lines. The cone is *circular* when its base, AXY ..., is a circle, and *right* if the perpendicular fall in the centre of the base.

Cor. 1. In similar cones [the pyramid is to be included] (486) the altitudes, radii vectores, like chords and generating lines for corresponding positions, are proportional; and the bases are as the squares of these lines.

Thus, $VP : Vp = PX : px = \text{chord}XY : \text{chd}xy = VX : Vx$,
and $\text{base}(AXY ...) : \text{bs}(axy ...) = (PX)^2 : (px)^2 = ...$

Cor. 2. If the vertex, V, be carried to an infinite distance, (487) the lines Aa, Xx, Yy, ..., will become parallel, and the figure axy ..., will $=$ AXY .... Under these conditions the solid (AXY ..., axy ...) is denominated, a *Cylinder* when the perimeter AXY ... is a curve, and a *Prism* when AXY ... is a polygon; and these magnitudes are said to be *right* or *oblique* according as the sides are perpendicular to or inclined to the bases. The cylinder and prism are also distinguished by their bases. When the base is a parallelogram it is obvious that all the other faces will be parallelograms and those opposite to each other equal, in which case the prism is called a *Parallelopipedon*; and the *Cube* is a right parallelopipedon of equal faces.

SECTION SECOND.

Surfaces of Solids.

PROPOSITION I.

The surface of a Polyhedron, that is, any solid bounded (488)
by planes, may be found by computing the areas of its several faces.

PROPOSITION II.

The convex surface of a Right Circular Cone is measured by its slant height multiplied into the semicircumference of its base. (489)

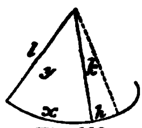


Fig. 112.

Let y be the convex surface included between any two positions of its *Generatrix*, l, and x the intercepted portion of the circular base; then, since y is obviously a continuous function of x, giving to y, x, the vanishing increments $[k]$, $[h]$, we have, (311), (148),

$$[k] = \tfrac{1}{2}l[h];$$

$$\therefore \qquad y' = \left[\frac{k}{h}\right] = \tfrac{1}{2}l = \tfrac{1}{2}lx^0;$$

$$\therefore (246)^* \qquad y = \tfrac{1}{2}lx;$$

$$\therefore \qquad y_{x=circumference} = \tfrac{1}{2}l(\textit{circumference of base}). \quad \text{Q. E. D.}$$

Cor. 1. The circular conical sector is measured by its (490)
slant height multiplied into half its base. $y = \frac{1}{2}lx$.

Cor. 2. The frustrumental surface of the right circular (491)
cone is measured by its slant height multiplied into the half sum of its bases.

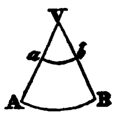


Fig. 112₂.

For let VAB, Vab, be conical sectors having the same vertical angle, V, and, consequently ABba the frustrumental surface in question, we have

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* No constant is to be added, since y and x vanish together.

$$\begin{aligned}ABba &= VAB - Vab = VA \cdot \tfrac{1}{3}AB - Va \cdot \tfrac{1}{3}ab \\ &= (Va + aA) \cdot \tfrac{1}{3}AB - Va \cdot \tfrac{1}{3}ab \\ &= aA \cdot \tfrac{1}{3}AB + Va \cdot \tfrac{1}{3}(AB - ab) \\ &= \frac{aA(AB + ab)}{2};\end{aligned}$$

since $\qquad \dfrac{VA}{Va} = \dfrac{AB}{ab}$, and $\therefore \dfrac{VA - Va}{Va} = \dfrac{AB - ab}{ab}$,

or $\quad Va = \dfrac{ab \cdot aA}{AB - ab}$.

Cor. 3. The convex surface of the Right Circular Cylinder, is equal to its height, multiplied into the circumference of its base ; for ab becomes $= AB$. (492)

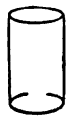

Fig. 112₃.

PROPOSITION III.

If a continuous curve referred to rectangular coördinates, revolve about the axis of abscissas, the derivative of the surface thus generated, regarded as a function of the corresponding abscissa, will be equal to the circumference described by the ordinate multiplied into the square root of unity increased by the square of the derivative of the ordinate also regarded as a function of the abscissa. (493)

For let the surface Z be described by the revolution of any continuous curve, z, around the axis, x, and M, m, k, h, the vanishing increments of Z, z, y, x;

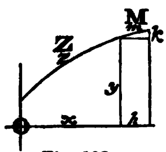

Fig. 113.

then $\quad Z'_{z=Fx} = \left[\dfrac{M}{h}\right]$,

but (491), $\quad M = m[\pi y + \pi(y + k)]$,

$$= (h^2 + k^2)^{\frac{1}{2}} \cdot \pi(2y + k);$$

$$\therefore \quad \left[\frac{M}{h}\right] = \frac{(h^2 + k^2)^{\frac{1}{2}}}{h} \cdot \pi \cdot 2y = 2\pi y\left(1 + \left[\frac{k}{h}\right]^2\right)^{\frac{1}{2}},$$

or, $\quad Z_{z=Fx} = 2\pi y(1 + y'^2_{y=fx})^{\frac{1}{2}}$. Q. E. D.

It follows that, in order to determine the surface generated by a particular curve, we have only to eliminate y and y' by aid of its equation, and to return to the function.

PROPOSITION IV.

A Spherical Zone is equal to its altitude multiplied into the circumference of the sphere. (494)

Let Z be a zone generated by the arc, z, of a circle revolving about a diameter, which we will assume as the axis of x, the origin being at the centre and the radius $= r$. We have

$$y^2 + x^2 = r^2,$$
$$(y+k)^2 + (x+h)^2 = r^2$$
$$\therefore \quad 2yk + k^2 + 2xh + h^2 = 0,$$
$$\frac{k}{h} = -\frac{2x+h}{2y+k};$$
$$\therefore \quad y'_{y=(r^2-x^2)^{\frac{1}{2}}} = -\frac{x}{y}, \text{ and } y'^2 = \frac{x^2}{y^2};$$
$$\therefore \quad Z'_{x-y_x} = 2\pi y\left(1+\frac{x^2}{y^2}\right)^{\frac{1}{2}} = 2\pi(y^2+x^2)^{\frac{1}{2}} = 2\pi r;$$
$$\therefore \quad Z = 2\pi r x,$$

where there is no constant to be added if we make the surface begin at the axis of y. Now let Z_1, x_1, Z_2, x_2, be corresponding values of Z and x; we have

$$\text{zone}(Z_2 - Z_1) = \text{zone}Z_2 - \text{zone}Z_1 = (x_2 - x_1) \cdot \pi \cdot 2r. \quad \text{Q. E. D.}$$

Cor. 1. A spherical zone is equal to the convex surface of the circumscribing cylinder, described by the revolution of a rectangle with a radius equal to that of the sphere. (495)

Surface described by S = sur. described by C.

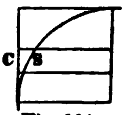

Fig. 114.

Cor. 2. The surface of the sphere is equal to the convex surface of the circumscribing cylinder, (496)

Or, to four *Great Circles*.

Sur. sphere $= 2Z_{x=r} = 4\pi r^2$.

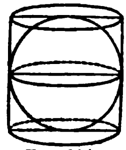
Fig. 114_2.

Cor. 3. The surfaces of spheres are to each other as the squares of their radii [r], or diameters [$2r$], or circumferences [$\pi \cdot 2r$]. (497)

EXERCISES.

1°. What will be the expense of gilding a globe 5 ft. in diameter at \$1·50 per square foot?

2°. What is the surface of the earth and of each of its zones, reckoning it as a sphere of 8000 miles in diameter, and the Obliquity of the Ecliptic at 23° 28′?

SECTION THIRD.

Volumes.

PROPOSITION I.

Rectangular Parallelopipedons are to each other as the products of their three dimensions. (498)

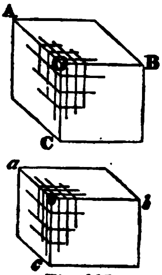

Fig. 115.

First, suppose their corresponding edges OA, OB, OC, *oa*, *ob*, *oc*, to be commensurable; that is, that OA being divided into m equal parts, oa contain an exact number, m', of the same parts, or that

$OA = mx$, $oa = m'x$, [x = the com. measure]
and $OB = ny$, $ob = n'y$, [y = measure of OB & ob]
and $OC = rz$, $oc = r'z$;

then will the partial rectangular parallelopipedons, formed by passing planes through the points of division parallel to the faces AB, AC, BC, *ab*, *ac*, *bc*, be all equal, since any one will be capable of superposition upon any other (487). It follows that the solids OABC, *oabc*, will contain severally mnr, $m'n'r'$, partial and equal rectangular parallelopipedons, and will consequently be to each other as mnr to $m'n'r'$;

$$\therefore \quad \frac{OABC}{oabc} = \frac{mnr}{m'n'r'} = \frac{mx \cdot ny \cdot rz}{m'x \cdot n'y \cdot r'z} = \frac{OA \cdot OB \cdot OC}{oa \cdot ob \cdot oc},$$

and the proposition is proved for the case in which the corresponding edges are commensurable.

Next, let the dimensions OA, OB, OC, *oa*, *ob*, *oc*, be any whatever, and put

$$OA = A,\ OB = B,\ OC = C;\ oa = a,\ ob = b,\ oc = c.$$

Now, if we increase a, b, c, by x, y, z, so as to become commensurable with A, B, C, and construct the parallelopipedon on $(a+x)$, $(b+y)$, $(c+z)$, from what has already been proved, there results

$$\frac{\text{parallelopipedon } [(a+x), (b+y), (c+z)]}{\text{parallelopipedon } [A, B, C]} = \frac{(a+x)(b+y)(c+z)}{ABC};$$

or $$\frac{\text{par'dn } [a, b, c]}{\text{par'dn } [A, B, C]} + \frac{\text{solid } [x, y, z]}{\text{par'dn } [A, B, C]} = \frac{abc}{ABC} + \frac{\dots}{ABC};$$

$\therefore$ (63), pard'n $[A, B, C]$: par'dn $[a, b, c]$: : ABC : abc.

Q. E. D.

Cor. 1. The rectangular parallelopipedon is measured by the product of its three dimensions, provided the *cube*, whose edge is the linear unit, be assumed as the unit of solidity. (499)

For, from par'dn $[a, b, c] = 1$, $a = b = c = 1$,
we have par'dn $[A, B, C] = ABC$.

Cor. 2. The right prism with a right angled triangular base, is measured by its base multiplied into its altitude. (500)

For the diagonal plane divides the rectangular parallelopipedon into two rectangular prisms, capable of superposition.

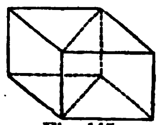

Fig. 115_2.

Cor. 3. Any right prism with a triangular base is equal to its base multiplied into its altitude. (501)

For the prism may be split into two, having right angled triangles for bases.

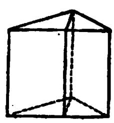

Fig. 115_3.

Cor. 4. Any right prism is measured by the product of its base and altitude. (502)

For the solid may be divided into prisms, having triangular bases.

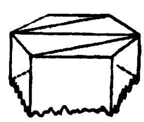

Fig. 115_4.

Cor. 5. The right cylinder is measured by the product of its base and altitude. (503)

For (502) is obviously independent of the number and magnitude of the sides.

PROPOSITION II.

The pyramid is equal to one-third of the product of its base and altitude, and the cone has a like measure. (504)

In the first place, suppose the pyramid, y, to have a triangular base, z, to which one of the edges, x, is perpendicular; and, for the purpose of finding the function $y = fx$, give to y, x, z, the corresponding increments k, h, i. Since the prisms constructed with the altitude h, and upon the bases $z, z+i$, will be inscribed in and circumscribe the solid, k, we have

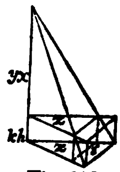

Fig. 116.

$$k \gtrless {}^{zh}_{(z+i)h}, \quad \therefore \quad \frac{k}{h} \gtrless {}^{z}_{z+i}, \quad \therefore \quad y'_{y=fx} = \left[\frac{k}{h}\right] = z \doteq ax^2; \quad (486)$$

$$\therefore \quad y = a \cdot \tfrac{1}{3}x^3 = \tfrac{1}{3}x \cdot ax^2 = \tfrac{1}{3}xz,$$

where no constant is to be added, because $y_{x=0} = 0$, and the proposition is proved for this particular case.

Next, let a right angled triangle, revolving about its perpendicular, p, and its base, varying in any way whatever, describe, the one a cone or pyramid, y, the other its base, x; giving to y and x the vanishing increments, k and h, from what has just been proved, we find

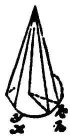

Fig. 116₂.

$$[k] = \tfrac{1}{3}p[h], \quad \therefore \quad y' = \tfrac{1}{3}p;$$

$$\therefore \qquad y = \tfrac{1}{3}px;$$

and the proposition is demonstrated for all cones and pyramids, in which the perpendicular falls within the base.

Lastly, let the base, u, be any whatever, and the perpendicular, p, fall upon its production; then, joining the foot of p, with two points of a contiguous portion of the perimeter so as to form the base, v, of a second pyramid or cone (p, v), we have

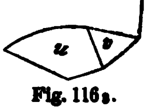

Fig. 116₃.

$$\text{cone } (p, u+v) = \tfrac{1}{3}p(u+v),$$

and $$\text{cone } (p, v) = \tfrac{1}{3}pv;$$

$$\therefore \qquad \text{cone } (p, u) = \tfrac{1}{3}pu. \quad \text{Q. E. D.}$$

PROPOSITION III

The Frustrum of a cone or pyramid is measured by one-third of the product of its altitude, multiplied into the sum of its bases augmented by a mean proportional between them. (505)

Let the base	$AXY \ldots = A$,	(*fig.* 111)
and	$axy \ldots = B$;	
the altitude,	$VP = x + a$,	
and	$Vp = x$;	

there results (504),

$$\text{solid } [(x+a),A] = \tfrac{1}{3}(x+a)A,$$

$$\text{solid } [x,B] = \tfrac{1}{3}xB;$$

$$\therefore \quad \text{frustrum } [A,B] = \tfrac{1}{3}(x+a)A - \tfrac{1}{3}xB = \tfrac{1}{3}[aA + x(A-B)];$$

but, $$\frac{A}{B} = \frac{(x+a)^2}{x^2}, \quad \therefore \quad \frac{x+a}{x} = \left(\frac{A}{B}\right)^{\frac{1}{2}},$$

$$\therefore \quad x = \frac{aB^{\frac{1}{2}}}{A^{\frac{1}{2}} - B^{\frac{1}{2}}}, \quad \therefore \quad x(A-B) = \frac{aB^{\frac{1}{2}}}{A^{\frac{1}{2}} - B^{\frac{1}{2}}}(A^{\frac{1}{2}} + B^{\frac{1}{2}})(A^{\frac{1}{2}} - B^{\frac{1}{2}});$$

$$\therefore \quad \text{frustrum } [A,B.a] = \tfrac{1}{3}a[A + (AB)^{\frac{1}{2}} + B]. \tag{505}$$

Cor. The prism or cylinder, whether right or oblique, is (506)
measured by the product of its base and altitude.

For $B = A$, gives $\tfrac{1}{3}a[A + (AB)^{\frac{1}{2}} + B] = Aa$.

PROPOSITION IV.

If a solid, V, *be generated by the motion of a plane,* U, (507)
varying according to the law of continuity, and remaining constantly similar to itself, and perpendicular to the axis of x; *then will the derivative of* V, *regarding* V *as a function of* x, *be equal to the generating plane,* U, *also regarded as a function of* x, *or*

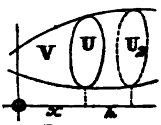

Fig. 117

$$V'_{V=Fx} = U = fx.$$

For, from a little reflection, it will be evident that the incremental solid $[U, U_2, h]$ must be measured as in (505), or that

$$V'_{V=Fx} = \left[\frac{[U, U_2, h]}{h}\right] = \tfrac{1}{3}[U + (UU_2)^{\frac{1}{2}} + U_2]$$

$$= \tfrac{1}{3}[fx + \{(fx \cdot f(x+h)\}^{\frac{1}{2}} + f(x+h)]$$

$$= \tfrac{1}{3}[fx + fx + fx] = fx = U. \quad \text{Q. E. D.}$$

Cor. 1. For any solid, generated by the revolution of a (508)
curve about the axis of x, the ordinate, y, describing the plane, U, we have

$$V'_{x-F_x} = U = \pi y^2 = \pi(fx)^2.$$

Cor. 2. For any volume embraced between the surfaces (509) described by the revolution of two curves, or the two branches of the same curve, U being described by the difference of the corresponding ordinates, y, y_2, we have

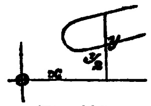

Fig. 117_2.

$$V'_x = \pi y^2 - \pi y_2^2 = \pi(y^2 - y_2^2) = \pi(y + y_2)(y - y_2).$$

Cor. 3. For the ellipsoidal frustrum, estimated from its equator, or from $x = 0$, we find

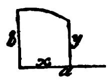

Fig. 117_3.

$$V_e = \pi \cdot \frac{b^2}{a^2}(a^2x - \tfrac{1}{3}x^3), \tag{510}$$

since, $\quad V'_x = \pi y^2 = \pi \cdot \dfrac{b^2}{a^2}(a^2 - x^2).$

Cor. 4. The corresponding frustrum of the circumscribing sphere, is

$$V_c = \pi(a^2x - \tfrac{1}{3}x^3), \tag{511}$$

since (510) becomes (511) when $b = a$.

Cor. 5. Prolate Ellipsoid $= 2V_{x-a} = \frac{4}{3}\pi ab^2$ (512)

$= \frac{2}{3} \cdot 2a \cdot \pi b^2 = \frac{2}{3}$(circumscribed cylinder)

$= 2$(inscribed double cone).

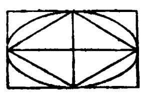
Fig. 117_4.

Cor. 6. $\text{Sphere}_{\text{radius}=a} = \frac{4}{3}\pi a^3 = \frac{2}{3} \cdot 2a \cdot \pi a^2$ (513)

$= \frac{2}{3}$(circumscribed cylinder)

$= 2$(inscribed double cone)

$= \frac{1}{3}a \cdot 4\pi a^2 = \frac{1}{3}a$(surface of sphere).

Scholium. The last relation might have been found by imagining the sphere filled with pyramids, having their common vertex at its centre, and their bases resting on its surface.

Cor. 7. The prolate ellipsoid and its circumscribing (514) sphere, and their frustrums corresponding to the same abscissa, are to each other as the square of the minor to the square of the major axis.

Prolate ellipsoid : $\text{sphere}_a = V_e : V_c = b^2 : a^2$.

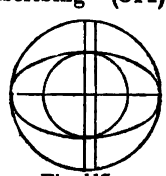
Fig. 117_5.

Cor. 8. Analogous relations may be found for the *Ob-* (515)

late Ellipsoid, described by the revolution of the ellipse about its minor axis by changing a into b, and b into a.

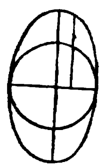

Fig. 117₆.

Thus $V_{2,e} = \pi \cdot \frac{a^2}{b^2}(b^2x - \frac{1}{3}x^3)$, $V_{2,c} = \pi(b^2x - \frac{1}{3}x^3)$;

$\therefore$ Oblate Ellipsoid $= \frac{2}{3} \cdot 2b \cdot \pi a^2 = \frac{2}{3}$(circumsc. cyl.)

$= 2 \cdot \frac{1}{3} \cdot 2b \cdot \pi a^2 = 2$(inscribed double cone);

Sphere$_{\text{radius}=b} = \frac{2}{3} \cdot 2b \cdot \pi b^2$

$\therefore$ Oblate Ellipsoid : Inscribed Sphere $= V_{2,e} : V_{2,s} = a^2 : b^2$.

Cor. 9. Common Paraboloid $V = \pi p x^2 = \frac{1}{2}x \cdot \pi y^2$ (516)

$= \frac{1}{2}$(circumscribing cylinder),

since $V' = \pi y^2 = \pi \cdot 2px$;

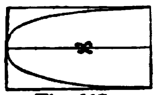

Fig. 117₇.

and, for the paraboloid whose equation is

$$y^2 = A_0 + A_1x + A_2x^2 + \dots,$$

we have $$V = \pi x(A_0 + \tfrac{1}{2}A_1x + \tfrac{1}{3}A_2x^2 + \dots) + V_{x=0}.$$

213

PROPOSITION V.

Similar solids are to each other as the cubes of their like dimensions. (517)

We understand by similar solids those in which all like dimensions are proportional; and, consequently, the sections through such dimensions similar.

The proposition becomes evident for the solids already investigated by making A and B vary as a^2 in (505), and b as a in (512) and (515), or by putting $A = ca^2$, $B = c_2a^2$, $b = c_3a$, the cs being constant, whence

Similar Cones, Pyramids, and their similar Frustrums, consequently similar Parallelopipedons, also similar Ellipsoids and Spheres

vary as $\frac{1}{3}[c + (cc_2)^{\frac{1}{2}} + c_2] \cdot a^3$, $\frac{4}{3}\pi c_3^2 \cdot a^3$,

or as a^3, or as the cubes of any like dimensions.

Next let any two similar perimeters, whether rectilinear or curvilinear, be similarly placed and revolve about any line OaA of corresponding radii vectores as an axis of x, the origin being at O; the ordinates y, y_2, of the extremities P, p, of any other corresponding radii vectores, will describe planes U, U_2, terminating (*fig.* 99.)

like frustrums of the two solids estimated from O where $x=0$, and this whether the radii y, y_2, remain constant for a given value of x or vary in any way whatever, that is, whether the perimeters ABC ..., *abc* ..., continue of the same magnitude or be variable, only that they preserve their similarity. Therefore we have

$$V' : V_2' = U : U_2 = y^2 : y_2^2 = x^2 : x_2^2;$$

$\therefore$ (507), $V : V_2 = \text{frustrum}[U] : \text{fr.}[U_2] = \frac{1}{3}x^3 : \frac{1}{3}x_2^3 = \text{OP}^3 : \text{O}p^3$, and the proposition is proved for this more general case.

Lastly, let the similar solids be any whatever, and assume any two like diameters for the axes of x, x_2, the origins dividing them proportionally. It follows from the definition of similar solids given above, that the generating planes U, U_2, perpendicular to x, x_2, will have corresponding positions in their respective solids V, V_2, when any diameter in U is to a like diameter in U_2 as the abscissa x terminating in U is to the abscissa x_2 terminating in U_2, and that U and U_2 will be similar figures;

$$\therefore \qquad U : U_2 = x^2 : x_2^2.$$

But from (300) it is manifest that (507) is applicable in this case also;

$$\therefore \qquad V' : V_2' = U : U_2 = x^2 : x_2^2,$$

and $V : V_2 = \text{fr.}[U] : \text{fr.}[U_2] = \frac{1}{3}x^3 : \frac{1}{3}x_2^3 = x^3 : x_2^3$. Q. E. D.

EXERCISES.

1°. A cylindrical cistern, capped with a hemispherical dome, is 15 feet deep and 10 in diameter. Required its capacity.

2°. A hollow cylinder, the side of which is one inch thick, is set into a cubical box, touching its sides and equalling it in height, and within the cylinder is placed a hollow sphere also an inch in thickness and tangent to the cylinder. What is the capacity of the sphere, it requiring just 10 gallons of water to fill the space between the box and the cylinder?

3°. What is the capacity of a cask, regarded as a frustrum of an ellipsoid, the bung diameter being 30 inches, the head diameters 25 each, and the length 40 inches; and how much will it differ from its inscribed double conical frustrum?

4°. What is the capacity of a paraboloidal cistern, having the dimensions in 1°?

5°. Wishing to ascertain the weight of a marble column 30 feet high, I take the semidiameters at the elevations 0, 10, 20, 30 feet, and find them to be 3, 4, 3, 2, feet; the specific gravity of marble is 2·7, water being 1, and a cubic foot of water weighs 1000 oz. avoirdupois. What is the weight?

6°. The earth may be regarded as an oblate spheroid, generated by the revolution of an ellipse about its shorter diameter of 7899·170 miles, while the equatorial diameter is 7925·648, according to Sir J. F. W. Herschel. Required the excess of volume over the inscribed sphere, and the quantity of water on the surface; allowing the sea to be to the land as 4 to 3, and its mean depth to be $2\frac{1}{4}$ miles.

7°. What must be the dimensions of a tub to hold 10 cubic feet, the depth and two diameters being as the numbers 4, 5, 6?

8°. What must be the dimensions of a paraboloidal kettle to contain 12 gallons, the diameter across the top being to the depth as 7 to 8?

BOOK SECOND.

SPHERICAL GEOMETRY.

SECTION FIRST.

Spherical Trigonometry.

PROPOSITION I.

A SPHERE, *being a solid bounded by a curve surface* (518) *everywhere equally distant from a point within, called the centre, may be generated by the revolution of a semicircle about its diameter.*

For, let the semicircumference, $PeEP'$, revolve about its diameter, $PoOP'$; it follows that every point, e, in $PeEP'$ will, while describing the circumference, $ee'e''$ ..., maintain a constant distance,

$$Oe = Oe' = Oe'' = \dots = OP = OP'$$
$$= OE = OE' = \dots,$$

from the centre, O; whence the surface described will be that of a sphere.

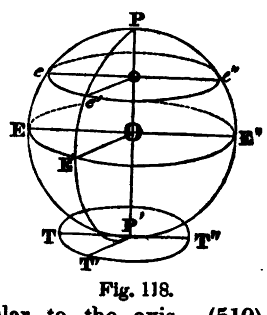

Fig. 118.

Cor. 1. The radius, OE, perpendicular to the axis, (519) POP′, will describe a *Great Circle*, the circumference of which, EE′E″ ..., denominated the *Equator*, will be everywhere 90° distant from its poles, P, P′.

Cor. 2. Any other perpendicular, oe, will describe a (520) *Small Circle*, also perpendicular to the same axis, and having the same poles, P, P′.

Cor. 3. Every section of a sphere by a plane is a circle. (521)

Cor. 4. The perpendicular, P′T, through the extremity (522)

of the diameter, POP′, will generate a plane (P′,TT′T″ ...) tangent to the sphere. Hence, a plane passing through the extremity of any radius, and perpendicular to it, is tangent to the sphere; and, conversely, a tangent plane is perpendicular to the radius, drawn to the point of contact.

Cor. 5. All great circles mutually bisect each other; as the *Meridians*, PEP′, PE′P′, since they have a common diameter, PP′. (523)

Cor. 6. Every great circle bisects the sphere. (524)

Cor. 7. A small circle divides the sphere into unequal parts, and is less the more distant it is from the centre. (525)

Cor. 8. A *Lune*, or the spherical surface embraced by two meridians, is to the whole surface of the sphere as its equatorial arc to the total equator; thus, (526)

$$\text{Lune PEP'E'P} : \text{Sph. Surface} = \text{arcEE'} : 360^\circ,$$

or $$\text{Lune}_{e,r} : 4\pi r^2 = e^\circ : 360^\circ;$$

$$\therefore\ \text{Lune}_{e,r} = \frac{e^\circ}{90^\circ}\cdot\pi r^2.\quad \begin{bmatrix} e = \text{eq.arc.} \\ r = \text{rad.sph.} \end{bmatrix} \tag{526}$$

Cor. 9. The *Spherical Wedge* or *Ungula*, PEP′E′O, is to the whole sphere as its equatorial arc is to the total equator; (527)

$$\text{Ungula}_{e,r} : \tfrac{4}{3}\cdot\pi r^3 = e^\circ : 360^\circ.$$

Cor. 10. Every meridian is perpendicular to its equator; as PE′P′ to EE′E″ .... (528)

Cor. 11. A *Spherical Angle* is identical with that embraced by the tangents to its sides, and is measured by the arc of its equator intercepted by these sides; as the angle EP′E′ = TP′T′ measured by EE′. (529)

Definition. A *Spherical Triangle* is the surface embraced by three arcs of great circles. (530)

PROPOSITION II.

A spherical triangle is equal to the sum of its three angles diminished by a semicircumference, multiplied into the square of the radius of the sphere. (531)

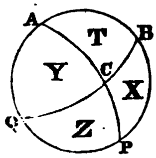

Fig. 119

Produce one side, AB, of the spherical triangle, *T*, so as to form the circumference, ABPQ, intersected in *P* and *Q* by the productions of the other sides, AC, BC; then, by the addition of the triangles, *X*, *Y*, *Z*, to *T*, there will be formed the lunes

$T+X$, $T+Y$, having the angles A, B, and to the surface $T+Z$, which may be readily shown by (523) to be equal to a lune with the angle, C;* and, from (526), there results,

$$T+X=\frac{A^\circ}{90^\circ}\cdot\pi r^2,\ T+Y=\frac{B^\circ}{90^\circ}\cdot\pi r^2,\ T+Z=\frac{C^\circ}{90^\circ}\cdot\pi r^2;$$

$$\therefore\ \frac{A^\circ+B^\circ+C^\circ}{90^\circ}\cdot\pi r^2=2T+(T+X+Y+Z)=2T+\frac{180^\circ}{90^\circ}\cdot\pi r^2;$$

$$\therefore\ T=\frac{A^\circ+B^\circ+C^\circ-180^\circ}{180^\circ}\cdot\pi r^2=(A+B+C-\pi)r^2. \quad (531)$$

Cor. 1. The sum of the three angles of a spherical triangle is always greater than two right angles and less than six; since for the existence of T we must obviously have (532)

$$A+B+C \begin{matrix} >\pi \\ <\pi+\pi+\pi \end{matrix}.$$

Cor. 2. Similar spherical triangles are to each other as the squares of the radii of their respective spheres, or as the squares of their homologous sides. (533)

Cor. 3. A spherical polygon is measured by the sum of its angles diminished by as many semicircumferences as it has sides save two, multiplied into the square of the radius of the sphere. (534)

For the number of sides being n, the polygon may be divided into $(n-2)$ triangles, T, T_2, T_3, ..., whose angles A, B, C; A_2, B_2, C_2; A_3, B_3, C_3; ..., make up the angles of the polygon, and there results

$$(T+T_2+T_3+\ldots)_{n-2}=[(A+B+C-\pi)+(A_2+B_2+C_2-\pi)+\ldots]r^2,$$
or $P_n=[S-(n-2)\pi]\cdot r^2$, $S=$ *sum of angles.*

* For, producing the arcs CP, CQ, till they meet in R, and drawing the diameters (?) AOP, BOQ, COR, we have only to show that the triangle PRQ, which completes the lune CPRQ, is equal to △ ACB. In order to this, take the point S, equally distant from A, B, C, or the pole of the small circle passing through those points, and draw the diameter SOT; then the arcs SA, SB, SC, TP, TQ, TR, will be all equal; also (?) ∠ ASB = PTQ, BSC = QTR, ASC = PTR, and it may therefore be shown by superposition that

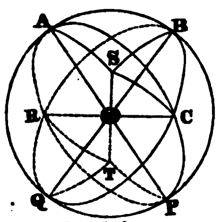

Fig. 119a.

$$\triangle QTP=ASB,\ RTQ=BSC,\ RTP=ASC;$$
$$\therefore\ \triangle PQR=QTP+RTQ-RTP$$
$$=ASB+BSC-ASC=ABC.\quad Q.\ E.\ D.$$

PROPOSITION III.

The cosine of any side of a spherical triangle is equal to the product of the cosines of the other two sides increased by that of their sines multiplied into the cosine of the angle opposite the first-mentioned side. (535)

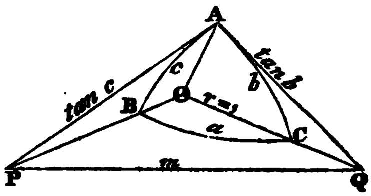

Fig. 120.

For, denoting the angles of the triangle by A, B, C, and the sides respectively opposite by a, b, c, let the radius of the sphere OA = OB = OC, be taken for unity, and draw the tangents, $\tan b$, $\tan c$, terminating in the productions of OC, OB, at Q, P, and join PQ $= m$; then will OQ $= \sec b$, OP $= \sec c$, and the spherical angle A will $= \angle$ PAQ, also $\angle$ BOC will be measured by a, (?), and from (388) there results,

$$\sec^2 b + \sec^2 c = m^2 + 2\sec b \sec c \cos a,$$

and
$$\tan^2 b + \tan^2 c = m^2 + 2\tan b \tan c \cos A;$$

$\therefore$
$$1 + 1 = 2\sec b \sec c \cos a - 2\tan b \tan c \cos A,$$

or
$$1 = \frac{1}{\cos b} \cdot \frac{1}{\cos c} \cdot \cos a - \frac{\sin b}{\cos b} \cdot \frac{\sin c}{\cos c} \cdot \cos A;$$

$$\left.\begin{aligned} &\text{whence} & \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ &\text{so} & \cos b &= \cos a \cos c + \sin a \sin c \cos B, \\ &\text{and} & \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} (535)$$

Cor. 1. If two spherical triangles have two sides of the one equal to two sides of the other each to each but the included angles unequal, the remaining sides will be unequal and the greater side will be opposite the greater angle; for, as A increases, $\cos A$ decreases and, consequently, $\cos a$ decreases or a increases, therefore A and a increase or decrease together. (536)

Cor. 2. Two spherical triangles are equal: (537)

1°. When two sides and the included angle of the one are respectively equal to the two sides and the included angle of the other; and therefore,

2°. When the sides of the one are severally equal to the sides of the other.

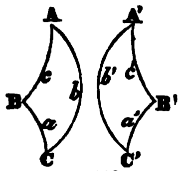

Fig. 120a.

For, 1°, if $b = b'$, $c = c'$, $A = A'$;

then $\quad a = a'$; $\therefore$ $B = B'$, $C = C'$;

$\therefore$ $\quad$ tr(ABC) = (A'B'C');

2°, if $\quad a = a'$, $b = b'$, $c = c'$;

then $\quad A = A'$; $\therefore$ tr(ABC = (A'B'C').

Cor. 3. The arc joining the vertex and the middle point (538) of the base of an isosceles spherical triangle, is perpendicular to the base, and bisects the vertical angle.

Cor. 4. We may adapt (535), which is the fundamental theorem of spherical trigonometry, to logarithmic computation, when a side is required, by putting

$$\sin b \cos A = \frac{\sin u}{\cos u} \cdot \cos b;$$

for then (535) becomes

$$\cos a = \cos b \cos c + \cos b \sin c \cdot \frac{\sin u}{\cos u}$$

$$= \frac{\cos b}{\cos u}(\cos c \cos u + \sin c \sin u);$$

and there results

$$\left.\begin{aligned} &\tan u = \tan b \cos A, \ \cos a = \frac{\cos b \cos(c-u)}{\cos u}; \\ \text{so} \quad &\tan v = \tan a \cos B, \ \cos b = \frac{\cos a \cos(c-v)}{\cos v}, \\ \text{and} \quad &\tan w = \tan a \cos C, \ \cos c = \frac{\cos a \cos(b-w)}{\cos w}. \end{aligned}\right\} \quad (539)$$

Cor. 5. When an angle is required, the transformations of Plane Trigonometry may be imitated, and we have

$$2\sin^2\tfrac{1}{2}A = 1 - \cos A$$

$$(535) \quad = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b \sin c}$$

$$= \frac{2\sin\frac{1}{2}(a+b-c) \cdot \sin\frac{1}{2}(a-b+c)}{\sin b \sin c}$$

$$\left.\begin{aligned}
\therefore\quad \sin\tfrac{1}{2}A &= \left[\frac{\sin\frac{1}{2}(a+b-c)\cdot\sin\frac{1}{2}(a+c-b)}{\sin b\,\sin c}\right]^{\frac{1}{2}}\\
&= \left[\frac{\sin(h-b)\,\sin(h-c)}{\sin b\,\sin c}\right]^{\frac{1}{2}};\\
\sin\tfrac{1}{2}B &= \left[\frac{\sin\frac{1}{2}(b+a-c)\cdot\sin\frac{1}{2}(b+c-a)}{\sin a\,\sin c}\right]^{\frac{1}{2}}\\
&= \left[\frac{\sin(h-a)\,\sin(h-c)}{\sin a\,\sin c}\right]^{\frac{1}{2}}\\
\text{and}\quad \sin\tfrac{1}{2}C &= \left[\frac{\sin\frac{1}{2}(c+a-b)\cdot\sin\frac{1}{2}(c+b-a)}{\sin a\,\sin b}\right]^{\frac{1}{2}}\\
&= \left[\frac{\sin(h-a)\,\sin(h-b)}{\sin a\,\sin b}\right]^{\frac{1}{2}}.
\end{aligned}\right\}\quad(540)$$

And by a like process we shall find

$$\left.\begin{aligned}
\cos\tfrac{1}{2}A &= \left[\frac{\sin h\,\sin(h-a)}{\sin b\,\sin c}\right]^{\frac{1}{2}}\\
\cos\tfrac{1}{2}B &= \left[\frac{\sin h\,\sin(h-b)}{\sin a\,\sin c}\right]^{\frac{1}{2}}\\
\cos\tfrac{1}{2}C &= \left[\frac{\sin h\,\sin(h-c)}{\sin a\,\sin b}\right]^{\frac{1}{2}}.
\end{aligned}\right\}\quad(541)$$

$$\left.\begin{aligned}
\therefore\quad \tan\tfrac{1}{2}A &= \left[\frac{\sin(h-b)\sin(h-c)}{\sin h\,\sin(h-a)}\right]^{\frac{1}{2}},\\
\tan\tfrac{1}{2}B &= \left[\frac{\sin(h-a)\sin(h-c)}{\sin h\,\sin(h-b)}\right]^{\frac{1}{2}},\\
\tan\tfrac{1}{2}C &= \left[\frac{\sin(h-a)\sin(h-b)}{\sin h\,\sin(h-c)}\right]^{\frac{1}{2}}.
\end{aligned}\right\}\quad(542)$$

Cor. 6. The sum of any two sides of a spherical triangle (543) is greater than the third side.

Cor. 7. If from any point within a spherical triangle, arcs (544) of great circles be drawn to the extremities of either side, the sum of the including sides will be greater than the sum of the included arcs. [See Plane Geometry.]

Cor. 8. A chain of spherical arcs is less the nearer it lies (545) to the arc joining its extremities.

Cor. 9. The arc of a great circle is the shortest distance (546) from point to point on a spherical surface.

Cor. 10. The sum of any two of the plane angles that (547) make up a solid angle, is greater than the third angle.

For, if about the solid angle, O, a sphere be described,

the plane angles	AOB, BOC, AOC,
will be measured by	AB, BC, AC,
the arcs of the spherical triangle	ABC.

Cor. 11. The sum of all the plane angles that make up (548) a solid angle, is less than four right angles.

For, let $a, b, c, \ldots, j, k, l$, be the arcs of the spherical polygon by which the plane angles are measured, and produce a, c, till they meet, forming the arcs, $a+m$, $c+n$; the new polygon whose sides are $a+m$, $c+n$, $d, \ldots, l$, will have a greater perimeter, consisting, however, of a number of sides less by one, than the original polygon. And this reduction may be continued till a triangle is obtained whose perimeter, less than a circumference, will be greater than that of the polygon.

Cor. 12. There can exist only (549)

Five Regular Polyhedrons.

Three—Tetrahedron, Octahedron, Icosahedron, whose faces are equilateral triangles, [$360^\circ : 60^\circ = 6$].

One—Hexahedron, squares, [$360^\circ : 90^\circ = 4$].

One—Dodecahedron, regular pentagons.

These figures may be formed of pasteboard, the lines of folding being cut half through

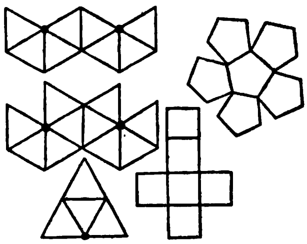

Fig. 120 s.

Cor. 13. If the three plane angles that constitute two (550) solid angles be respectively equal, the homologous planes will be equally inclined to each other.

PROPOSITION IV.

In any spherical triangle the sines of the sides are to (551) *each other as the sines of the angles respectively opposite.*

Taking the double products of (540) and (541), observing that $2\sin\frac{1}{2}A \cos\frac{1}{2}A = \sin A$, we have

$$\left.\begin{aligned}\sin A &= \frac{2[\sin h \sin(h-a)\sin(h-b)\sin(h-c)]^{\frac{1}{2}}}{\sin b \sin c},\\ \sin B &= \frac{2[\sin h \sin(h-a)\sin(h-b)\sin(h-c)]^{\frac{1}{2}}}{\sin a \sin c},\\ \sin C &= \frac{2[\sin h \sin(h-a)\sin(h-b)\sin(h-c)]^{\frac{1}{2}}}{\sin a \sin b};\end{aligned}\right\} \quad (552)$$

$$\therefore \quad \frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}, \quad \frac{\sin A}{\sin C} = \frac{\sin a}{\sin c}, \quad \frac{\sin B}{\sin C} = \frac{\sin b}{\sin c}. \quad \text{Q. E. D.}$$

Cor. 1. In a spherical triangle the greater side is opposite the greater angle, and *v. v.* (553)

Cor. 2. The perpendicular arc is the shortest distance from any given point to an arc also given in position. (554)

Cor. 3. The angles opposite the equal sides of an isosceles spherical triangle, are equal, and *vice versâ.* (555)

Cor. 4. An equilateral spherical triangle is equiangular, and *v. v.* (556)

PROPOSITION V.

To eliminate from equations (535) *the cosine of each side in succession.*

Substituting the value of $\cos c$ from the third in the second, we find

$$\begin{aligned}\cos b &= \cos a(\cos a \cos b + \sin a \sin b \cos C) + \sin a \sin c \cos B\\ &= \cos^2 a \cos b + \sin a \cos a \sin b \cos C + \sin a \sin c \cos B;\end{aligned}$$

$$\therefore \quad (1-\cos^2 a)\cos b = \sin a \cos a \sin b \cos C + \sin a \sin c \cos B$$

or $\quad \sin^2 a \cos b = \sin a \cos a \sin b \cos C + \sin a \sin c \cos B,$

and $\sin a \cos b = \cos a \sin b \cos C + \sin c \cos B,$

so

$$\left.\begin{aligned}\sin a \cos b &= \cos a \sin b \cos C + \sin c \cos B,\\ \sin a \cos c &= \cos a \sin c \cos B + \sin b \cos C,\\ \sin b \cos c &= \cos b \sin c \cos A + \sin a \cos C;\\ \sin b \cos a &= \cos b \sin a \cos C + \sin c \cos A,\\ \sin c \cos a &= \cos c \sin a \cos B + \sin b \cos A,\\ \sin c \cos b &= \cos c \sin b \cos A + \sin a \cos B.\end{aligned}\right\} \quad (557)$$

Note. Instead of going through independent operations, all the above forms may be obtained from the first by a simple change of letters; thus, changing b, B, into c, C, and *vice versâ*, the first becomes the second, which in turn gives the third by commuting a, A, and b, B, and so on.

PROPOSITION VI.

From (557) to eliminate a sine, in order that no more than two sides may be embraced in each equation.

From (551) we have $\frac{\sin c}{\sin b} = \frac{\sin C}{\sin B}$, or $\sin c = \frac{\sin b \sin C}{\sin B}$, which substituted in the first of (557), gives

$$\sin a \cos b = \cos a \sin b \cos C + \frac{\sin b \sin C}{\sin B} \cdot \cos B,$$

$$\therefore \quad \sin a \cdot \frac{\cos b}{\sin b} = \cos a \cos C + \frac{\cos B}{\sin B} \cdot \sin C;$$

$$\therefore \quad \left.\begin{aligned} \sin a \cot b &= \cos a \cos C + \cot B \sin C, \\ \sin a \cot c &= \cos a \cos B + \cot C \sin B, \\ \sin b \cot c &= \cos b \cos A + \cot C \sin A; \\ \sin b \cot a &= \cos b \cos C + \cot A \sin C, \\ \sin c \cot a &= \cos c \cos B + \cot A \sin B, \\ \sin c \cot b &= \cos c \cos A + \cot B \sin A. \end{aligned}\right\} \quad (558)$$

PROPOSITION VII.

From (558) to eliminate a side.

From the fourth and first by aid of (551) we have

$$\cot a = \frac{\cos b}{\sin b} \cdot \cos C + \cot A \sin C \cdot \frac{1}{\sin b}$$

$$= \cot b \cos C + \cot A \sin C \cdot \frac{\sin A}{\sin a \sin B},$$

and $$\cot b = \cot a \cos C + \cot B \sin C \cdot \frac{1}{\sin a};$$

$$\therefore \quad \cot a = \cot a \cos^2 C + \cot B \sin C \cos C \cdot \frac{1}{\sin a}$$

$$+ \cot A \sin C \cdot \frac{\sin A}{\sin A \sin B},$$

and $$\cot a(1 - \cos^2 C) = \frac{\cot B \sin C \cos C}{\sin a} + \frac{\cot A \sin A \sin C}{\sin a \sin B},$$

or $$\frac{\cos a}{\sin a} \cdot \sin^2 C = \frac{\cos B}{\sin B} \cdot \frac{\sin C \cos C}{\sin a} + \frac{\cos A}{\sin A} \cdot \frac{\sin A \sin C}{\sin a \sin B},$$

$$\therefore \quad \sin B \sin C \cos a = \cos B \cos C + \cos A;$$

$$\left.\begin{aligned} \therefore\ \cos(180^\circ-A) &= \cos B\cos C+\sin B\sin C\cos(180^\circ-a),\\ \cos(180^\circ-B) &= \cos A\cos C+\sin A\sin C\cos(180^\circ-b),\\ \cos(180^\circ-C) &= \cos A\cos B+\sin A\sin B\cos(180^\circ-c). \end{aligned}\right\}\ (559)$$

Cor. 1. If in these equations we substitute

$$\begin{aligned} A &= 180^\circ-\mathfrak{a}, & a &= 180^\circ-\mathfrak{A},\\ B &= 180^\circ-\mathfrak{b}, & b &= 180^\circ-\mathfrak{B},\\ C &= 180^\circ-\mathfrak{c}; & c &= 180^\circ-\mathfrak{C}; \end{aligned}$$

there results,

$$\begin{aligned} \cos\mathfrak{a} &= \cos\mathfrak{b}\cos\mathfrak{c}+\sin\mathfrak{b}\sin\mathfrak{c}\cos\mathfrak{A},\\ \cos\mathfrak{b} &= \cos\mathfrak{a}\cos\mathfrak{c}+\sin\mathfrak{a}\sin\mathfrak{c}\cos\mathfrak{B},\\ \cos\mathfrak{c} &= \cos\mathfrak{a}\cos\mathfrak{b}+\sin\mathfrak{a}\sin\mathfrak{b}\cos\mathfrak{C}; \end{aligned}$$

which are of the same form with (535), and therefore belong to a triangle whose sides are $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, and angles $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; the relation of these two triangles, ABC, $\mathfrak{ABC}$, is

$$\left.\begin{aligned} A+\mathfrak{a} &= 180^\circ = \mathfrak{A}+a,\\ B+\mathfrak{b} &= 180^\circ = \mathfrak{B}+b,\\ C+\mathfrak{c} &= 180^\circ = \mathfrak{C}+c; \end{aligned}\right\}\ (560)$$

that is, the sides of the one are *supplementary* to the *angles* of the other; hence they are denominated *Supplemental Triangles*, more commonly known as *Polar*, from their geometrical construction, which is as follows.

About the angles A, B, C, as poles, with a radius of 90°, describe the arcs $\mathfrak{BC}=\mathfrak{a}$, $\mathfrak{AC}=\mathfrak{b}$, $\mathfrak{AB}=\mathfrak{c}$, forming the spherical triangle $\mathfrak{ABC}$, polar to ABC; $\mathfrak{C}$, being the intersection of $\mathfrak{BC}$, $\mathfrak{AC}$, equators to A, B, is 90° distant from A, B, and therefore the pole of AB, and in the same way $\mathfrak{A}$, $\mathfrak{B}$, are shown to be the poles of BC, AC; $\therefore$ if two sides, as AB, AC, be produced to intersect $\mathfrak{BC}$ in m, n, we find

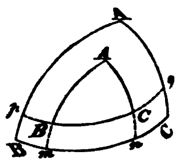

Fig. 121.

$$\begin{aligned} &\mathfrak{B}m+mn=90^\circ,\ mn+n\mathfrak{C}=90^\circ;\\ \therefore\quad &mn+(\mathfrak{B}m+mn+n\mathfrak{C})=180^\circ,\\ \text{or}\quad &A+\mathfrak{a}=180^\circ; \qquad [mn=\text{measure of }A]\\ \text{so}\quad &B+\mathfrak{b}=180^\circ,\\ \text{and}\quad &C+\mathfrak{c}=180^\circ;\\ \text{again}\quad &\mathfrak{A}+a=pq+BC=pC+Bq=180^\circ,\\ &\mathfrak{B}+b=180^\circ,\\ &\mathfrak{C}+c=180^\circ. \end{aligned}$$

Cor. 2. Instead of transforming (559) we may apply (560) to the polar triangle of (540), and

$$\sin\tfrac{1}{2}\mathfrak{A} = \left[\frac{\sin\frac{1}{2}(\mathfrak{a}+\mathfrak{b}-\mathfrak{c})\sin\frac{1}{2}(\mathfrak{a}+\mathfrak{c}-\mathfrak{b})}{\sin\mathfrak{b}\,\sin\mathfrak{c}}\right]^{\frac{1}{2}}$$

becomes $\sin\frac{1}{2}(180^\circ - a) =$

$$\left[\frac{\sin\frac{1}{2}[(180^\circ-A)+(180^\circ-B)-(180^\circ-C)]\sin\frac{1}{2}[180^\circ-(A+C-B)]}{\sin(180^\circ-B)\,\sin(180^\circ-C)}\right]^{\frac{1}{2}},$$

or $$\cos\tfrac{1}{2}a = \left[\frac{\cos\frac{1}{2}(A+B-C)\,\cos\frac{1}{2}(A+C-B)}{\sin B\,\sin C}\right]^{\frac{1}{2}};$$

or, putting $H = \frac{1}{2}(A+B+C)$,

$$\left.\begin{aligned}
\cos\tfrac{1}{2}a &= \left[\frac{\cos(H-B)\,\cos(H-C)}{\sin B\,\sin C}\right]^{\frac{1}{2}},\\
\cos\tfrac{1}{2}b &= \left[\frac{\cos(H-A\,\cos(H-C)}{\sin A\,\sin C}\right]^{\frac{1}{2}},\\
\cos\tfrac{1}{2}c &= \left[\frac{\cos(H-A)\,\cos(H-B)}{\sin A\,\sin B}\right]^{\frac{1}{2}};
\end{aligned}\right\}\quad(561)$$

so
$$\left.\begin{aligned}
\sin\tfrac{1}{2}a &= \left[\frac{-\cos H\,\cos(H-A)}{\sin B\,\sin C}\right]^{\frac{1}{2}},\\
\sin\tfrac{1}{2}b &= \left[\frac{-\cos H\,\cos(H-B)}{\sin A\,\sin C}\right]^{\frac{1}{2}},\\
\sin\tfrac{1}{2}c &= \left[\frac{-\cos H\,\cos(H-C)}{\sin A\,\sin B}\right]^{\frac{1}{2}};
\end{aligned}\right\}\quad(562)$$

$\therefore$
$$\left.\begin{aligned}
\tan\tfrac{1}{2}a &= \left[\frac{-\cos H\,\cos(H-A)}{\cos(H-B)\,\cos(H-C)}\right]^{\frac{1}{2}},\\
\tan\tfrac{1}{2}b &= \left[\frac{-\cos H\,\cos(H-B)}{\cos(H-A)\,\cos(H-C)}\right]^{\frac{1}{2}},\\
\tan\tfrac{1}{2}c &= \left[\frac{-\cos H\,\cos(H-C)}{\cos(H-A)\,\cos(H-B)}\right]^{\frac{1}{2}}.
\end{aligned}\right\}\quad(563)$$

Cor. 3. Applying to (559) a process like that for (539), we find

$$\left.\begin{aligned}
\tan U &= \tan B\cos(180^\circ-a),\ \cos(180^\circ-A) = \frac{\cos B\,\cos(C-U)}{\cos U},\\
\tan V &= \tan A\cos(180^\circ-b),\ \cos(180^\circ-B) = \frac{\cos A\,\cos(C-V)}{\cos V},\\
\tan W &= \tan A\cos(180^\circ-c),\ \cos(180^\circ-C) = \frac{\cos A\cos(B-W)}{\cos W}.
\end{aligned}\right\}\quad(564)$$

Cor. 4. Forms (383), (384), may be frequently applied to (535), (557), (558), and (559), with advantage. (565)

PROPOSITION VIII.

Napier's Analogies.

$$\left.\begin{aligned}\cos\tfrac{1}{2}(a+b) &: \cos\tfrac{1}{2}(a-b) = \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A+B),\\ \sin\tfrac{1}{2}(a+b) &: \sin\tfrac{1}{2}(a-b) = \cot\tfrac{1}{2}C : \tan\tfrac{1}{2}(A-B);\end{aligned}\right\}\ (566)$$

$$\left.\begin{aligned}\cos\tfrac{1}{2}(A+B) &: \cos\tfrac{1}{2}(A-B) = \tan\tfrac{1}{2}c : \tan\tfrac{1}{2}(a+b),\\ \sin\tfrac{1}{2}(A+B) &: \sin\tfrac{1}{2}(A-B) = \tan\tfrac{1}{2}c : \tan\tfrac{1}{2}(a-b).\end{aligned}\right\}\ (567)$$

Adding the first and fourth of (557), we get

$$\sin(a+b) = \sin(a+b)\cos C + (\cos A + \cos B)\sin c,$$

or $$(\cos A + \cos B)\sin c = \sin(a+b)\,(1-\cos C);$$

but (551), $$\sin A \sin c = \sin a \sin C,$$
$$\sin B \sin c = \sin b \sin C;$$

$$\therefore\quad (\sin A + \sin B)\sin c = (\sin a + \sin b)\sin C,$$
$$(\sin A - \sin B)\sin c = (\sin a - \sin b)\sin C;$$

$$\therefore\quad \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\sin a + \sin b}{\sin(a+b)} \cdot \frac{\sin C}{1-\cos C},$$

$$\therefore\quad \frac{\sin A - \sin B}{\cos A + \cos B} = \frac{\sin a - \sin b}{\sin(a+b)} \cdot \frac{\sin C}{1-\cos C};$$

and by (335), (329), (321), (330), (321), $(336)_{r=0}$, the last two forms become

$$\left.\begin{aligned}\therefore\quad \tan\tfrac{1}{2}(A+B) &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cdot\cot\tfrac{1}{2}C,\\ \tan\tfrac{1}{2}(A-B) &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cdot\cot\tfrac{1}{2}C;\end{aligned}\right\}\ (566)$$

$$\left.\begin{aligned}\therefore (560)\quad \tan\tfrac{1}{2}(a+b) &= \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\cdot\tan\tfrac{1}{2}c,\\ \tan\tfrac{1}{2}(a-b) &= \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\cdot\tan\tfrac{1}{2}c.\end{aligned}\right\}\ (567)$$

The above proportions (566), (567), should be expressed in words, and committed to memory; since they make known by easy logarithmic operations, 1°, when two sides and the included angle are given, the remaining angles; and, 2°, when two angles and the included side are the data, the remaining sides.

PROPOSITION IX.

Napier's Rules.

If, in a right angled spherical triangle, we denominate

CIRCULAR PARTS,

The side opposite the right angle with the including angles, and the complements of the other two sides, then

$$\textit{Cos. Mid. Part} = \textit{Product of Sines of Opp. Parts} \qquad (568)$$
$$= \textit{Product of Cotangents of Adj. Parts},$$

observing that each of the five parts will be adjacent to two and opposite to two, no account being made of the right angle.

In order to demonstrate this proposition, which is a most remarkable example of artificial memory, it is only necessary to make $A = 90°$ and seek among the preceding forms, values for

$$a, B, C; \; 90° - b = b_c,^* \; 90° - c = c_c;$$

we find

$$\cos a = \cos b \cos c, \text{ or } \cos B \cos C = \sin B \sin C \cos a;$$
$$\cos B = \sin C \cos b, \text{ or } \sin c \cos a = \cos c \sin a \cos B;$$
$$\cos C = \sin B \cos c, \text{ or } \sin b \cos a = \cos b \sin a \cos C;$$
$$\sin b = \sin a \sin B, \text{ or } \sin b \cot c = \cot C;$$
$$\sin c = \sin a \sin C, \text{ or } \sin c \cot b = \cot B;$$

and these reduce to

$$\left.\begin{aligned}
\cos a &= \sin b_c \sin c_c = \cot B \cot C,\\
\cos B &= \sin b_c \sin C = \cot a \cot c_c,\\
\cos C &= \sin c_c \sin B = \cot a \cot b_c,\\
\cos b_c &= \sin a \sin B = \cot c_c \cot C,\\
\cos c_c &= \sin a \sin C = \cot b_c \cot B.
\end{aligned}\right\} (568)$$

Cor. 1. Napier's Rules may be employed in solving (569) *Quadrantal Triangles*, by observing that the circular parts will be the supplement of the angle opposite the quadrant, with the sides adjacent, and the complements of the other two angles.

For, let a = quadrant = $90°$,

then (560), $\qquad \mathfrak{A} = 180° - a = 90°$,

and the triangle $\mathfrak{ABC}$, polar to ABC, being right angled at $\mathfrak{A}$, gives (568)

$$\cos \mathfrak{a} = \sin \mathfrak{b}_c \sin \mathfrak{c}_c = \cot \mathfrak{B} \cot \mathfrak{C},$$

$$\therefore (560) \cos(180° - A) = \sin[90° - (180° - B)] \sin(C - 90) = \ldots,$$

or, putting $180° - A = A_{\prime}$, &c., and reducing,

---

* May be read the complement of b, or b complement.

$$\left.\begin{aligned} &\cos A_e = \sin B_c \sin C_c = \cot B \cot C\,; \\ \text{so}\qquad &\cos b = \sin c \sin B_c = \cot A_e \cot C_c, \\ &\cos c = \sin b \sin C_c = \cot A_e \cot B_c, \\ &\cos B_e = \sin A_e \sin b = \cot C_c \cot c, \\ &\cos C_c = \sin A_e \sin c = \cot B_c \cot b. \end{aligned}\right\}(569)$$

It will be observed that the quadrant, a, is regarded as not separating B, C.

For illustration, let it be required to find B when a and c are given, and $b = 90°$.

We have $\qquad \cos(180° - B) = \cos B_e = \cot a \cot c.$

Cor. 2. Napier's Rules may be employed in the solution (570) of *Isosceles Spherical Triangles*, by observing that the circular parts will be the equal side, equal angle, half unequal angle, and complement of half unequal side.

For, let $c = a$; then joining B and the middle point of b, we shall have formed two right angled triangles, in each of which the parts will be (538), (568),

$$a,\ A,\ \tfrac{1}{2}B,\ (\tfrac{1}{2}b)_c\,;$$

$$\therefore\qquad \left.\begin{aligned} &\cos a = \cot A \cot\tfrac{1}{2}B, \\ &\cos A = \cot a \cot(\tfrac{1}{2}b)_c, \\ &\cos(\tfrac{1}{2}b)_c = \sin a \sin\tfrac{1}{2}B, \\ &\cos\tfrac{1}{2}B = \sin A \sin(\tfrac{1}{2}b)_c\,. \end{aligned}\right\}(570)$$

Cor. 3. Napier's Rules may be employed in the solution (571) of *Oblique Triangles*, by dropping a perpendicular so that two of the known parts shall fall in one of the right angled triangles thus formed; there will result also,

Bowditch's Rules.

1°. The cosines of the parts opposite the perpendicular, are proportional to the sines of those adjacent.

2°. The cosines of the parts adjacent to the perpendicular, are proportional to the cotangents of the parts opposite.

For, let the angle, C, be separated into the parts, M, N, by the perpendicular, p, dividing the opposite side, c, into the corresponding parts, m, n, subjacent to a, b, and we find,

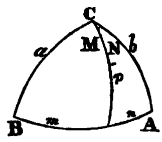

Fig. 121$_2$.

$$\left.\begin{aligned} &\cos a = \sin p_c \sin m_c,\ \cos b = \sin p_c \sin n_c\,; \\ &\cos B = \sin p_c \sin M,\ \cos a = \sin p_c \sin N\,; \end{aligned}\right\}(1°)$$

$$\left.\begin{aligned} &\cos m_c = \cot p_c \cot B,\ \cos n_c = \cot p_c \cot A\,; \\ &\cos M = \cot p_c \cot a,\ \cos N = \cot p_c \cot b. \end{aligned}\right\}(2°)$$

Cases in Spherical Trigonometry.

Given.		Forms for Solution.
I.	Two sides and included angle.	(535) ; (539) ; (566) ; (571).
II.	Two angles and included side.	(559) ; (564) ; (567) ; (571).
III.	Two sides and opposite angle.	(535) and (384) ; (551), [?].
IV.	Two angles and opposite side.	(559) and (384) ; (551).
V.	Three sides.	(535); (540); (541); (542); (552).
VI.	Three angles.	(559) ; (561) ; (562) ; (563).
Right Triangles.		(568).
Quadrantal Triangles.		(569).
Isosceles Triangles.		(570).

Scholium I. There will be a choice in forms, not only on account of logarithmic operations, but also for the purpose of avoiding the cosines of very small arcs, and the sines of those differing little from a quadrant.

Scholium II. Problems pertaining to spherical trigonometry will generally find their easiest solution by constructing the triangle so that one of its angles shall be at the pole of the sphere.

Latitudes and Longitudes.

Places.	Latitude.	Longitude.
Boston (State House), Mass.,	42°21′22·7″N.	71° 4′ 9″W.
Chicago, Ill.,	42 0 0 N.	87 35 W.
Canton, China,	23 8 9 N.	113 16 54 E.
Cape Good Hope (Obs.), Africa,	33 56 3 S.	18 28 45 E.
Cape Horn, S. America,	55 58 41 S.	67 10 53 W.
Cincinnati (Fort Washington), Ohio,	39 5 54 N.	84 27 0 W.
Greenwich (Obs.), Eng.,	51 28 39 N.	0 0 0
New York (City Hall), N. Y.,	40 42 40 N.	74 1 8 W.
Paris, (Obs.), France,	48 50 13 N.	2 20 24 E.
Philadelphia (Ind'ce Hall), Pa.,	39 56 59 N.	75 9 54 W.
Rome (Roman Col.), Italy,	41 53 52 N.	12 28 40 E.
Washington (Capitol), D. C.,	38 53 23 N.	77 1 24 W.

1°. Required the distance and direction from New York to Greenwich. [69·2 miles to 1°.]

$$\begin{aligned}\text{Polar distance of New York} &= 49^\circ\ 17'\ 20'',\\ \text{Polar distance of Greenwich} &= 38^\circ\ 31'\ 21'';\\ \text{Difference of longitude} &= 74^\circ\ 1'\ 8'';\end{aligned}$$

the problem, therefore, belongs to Cases I., III.

$\therefore$	(539),	$\tan u = \tan 49^\circ\ 17'\ 20''$	0.065262
		$\times \cos 74^\circ\ 1'\ 8''$	$\bar{1}.439838$
$\therefore$		$u = 17^\circ\ 44'\ 34''$	$\bar{1}.505100$
$\therefore$		$\cos a = \cos 49^\circ\ 17'\ 20''$	$\bar{1}.814411$
		$\times \cos 20^\circ\ 46'\ 47''$	$\bar{1}.970789$
		$: \cos 17^\circ\ 44'\ 34''$	$\bar{1}.978835$
$\therefore$		$a = 3473$ miles,	$\bar{1}.806365$
$\therefore$	(551),	$\sin 50^\circ\ 11'\ 17''$	$\bar{1}.885446$
		$: \sin 74^\circ\ 1'\ 8''$	$\bar{1}.982883$
		$= \sin 38^\circ\ 31'\ 21''$	$\bar{1}.794363$
		$: \sin 51^\circ\ 12'\ 43''$	$\bar{1}.891800$

Solve the same by (566), also by (571).

2°. Required the distances between New York, Cincinnati and Washington, the angles of the spherical triangle thus constructed, and its area.

3°. Required the distance and bearing of Chicago from Boston, reckoning the latitudes of both places at 42°.

4°. Required the direction and distance from Greenwich to Quito, the latter place being under the equator, and having 79° W. lon., nearly.

5°. Required the breadth of South America between Cape Blanco and Cape St. Roque, reckoning both to have 4° 30′ S. lat., and a difference of longitude = 46°.

6°. To reduce an angle to the horizon. (572)

Around the point of observation, O, imagine a sphere to be described with radius = 1; then will one of its arcs, a, be the measure of the inclined angle, SOS′, and the other two, b, c, of the angles, S′OP, SOP, made with the vertical, OP; whence the spherical angle, A, opposite a, will be equal to the horizontal angle required.

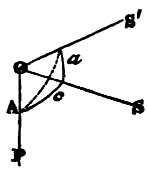

Fig. 121 s.

For example, let $a = 36{\cdot}7^\circ$, $b = 110^\circ$, $c = 75{\cdot}25^\circ$; then will A be found $= 12^\circ\ 0'\ 20''$.

7°. Through any two given points and a third upon the surface of a sphere, which do not lie in the circumference of the same great circle, there may be made to pass two equal and parallel small circles; that is, one of them through the first two given points, and the other through the third given point; and every spherical arc which is terminated by these circles shall be bisected by the circumference of the great circle, to which they are parallel.*

8°. If there be two equal and parallel small circles, and if a great circle meets one of them in any point, it will meet the other in the opposite extremity of the diameter which passes through that point.

9°. If a great circle cuts one of two equal and parallel small circles, it will cut the other likewise; also, if it touches one of them, it will touch the other likewise.

10°. Lunular portions of surface, which are contained by equal spherical arcs with the arcs of equal small circles of the same sphere, are equal to one another; so also are the pyramidal solids, which have these portions for their bases, and their common vertex in the centre of the sphere.

11°. Spherical triangles, which stand upon the same base and between the same equal and parallel small circles, are equal to one another.

12°. If equal triangles, ABC, EBC, stand upon the same base, BC, and the same side of it, the points, A, E, and B, C, lie in the circumference of two equal and parallel small circles.

13°. Of equal spherical triangles upon the same base, the isosceles has the least perimeter.

14°. Of all triangles which are upon the same base, and have equal perimeters, the isosceles has the greatest area.

15°. If two spherical triangles have two sides of the one equal to the two sides of the other, each to each, and the angle which is contained by the two sides of the first equal to the sum of the other two angles of that triangle, but the angle which is contained by the two sides of the other not so; the first triangle shall be greater than the other.

16°. Two given finite spherical arcs, together with a third inde-

* This and the following exercises, drawn from the Library of Useful Knowledge, may be performed at the student's leisure.

finite, inclose the greatest surface possible, when placed so that the included angle may be equal to the sum of the other two angles of the triangle.

17°. In a triangle, ABC, which has one of its angles, ABC, equal to the sum of the other two, the containing sides, AB, BC, are together less than a semicircumference.

18°. Of all spherical polygons, contained by the same given sides, that one contains the greatest portion of the spherical surface which has all its angles in the circumference of a circle.

19°. A circle includes a greater portion of the spherical surface than any spherical polygon of the same perimeter.

20°. The lunular surface, which is included by a spherical arc, and a small arc, is greater than any other surface which is included by the same perimeter, of which the same spherical arc is a part.

21°. Of all spherical polygons having the same number of sides and the same perimeter, the greatest is that which has all its sides equal and all its angles equal.

22°. Spherical pyramids, which stand upon equal bases, are equal to one another; so, likewise, are their solid angles.

23°. Any two spherical pyramids are to one another as their bases, and the solid angles of the pyramids are to one another in the same ratio.

24°. Every spherical pyramid is equal to the third part of the product of its base and the radius of the sphere.

25°. Every solid angle is measured by the spherical surface which is described with a given radius about the angular point, and intercepted between its planes.

26°. To find the diameter of a given sphere.*

27°. To find the quadrant of a great circle.

28°. Any point being given upon the surface of a sphere, to find the opposite extremity of the diameter which passes through that point.

29°. To join two given points upon the surface of a sphere.

30°. A spherical arc being given, to complete the great circle of which it is a part.

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* In executing this and the following problems, it is not permitted to employ any thing like a flexible ruler or straightedge, but the student is supposed to be furnished simply with a pair of compasses of such construction as to be capable of embracing the extremities of any arc not greater than a quadrant. A *Spherical Blackboard* would be found useful.

31°. To bisect a given spherical arc.

32°. To draw a spherical arc which shall bisect a given spherical arc at right angles.

33°. To draw an arc which shall be perpendicular to a given spherical arc, from a given point in the same.

34°. To draw an arc which shall be perpendicular to a given spherical arc, from a given point without it.

35°. To bisect a given spherical angle.

36°. At a given point in a given arc, to make a spherical angle equal to a given spherical angle.

37°. To describe a circle through three given points upon the surface of a sphere.

38°. To find the poles of a given circle.

39°. Through two given points, A, B, and a third point, C, on the surface of a sphere, to describe two equal and parallel small circles; the points A, B, C, not lying in the circumference of the same great circle.

40°. To describe a triangle which shall be equal to a given spherical polygon, and shall have a side and adjacent angle the same with a given side and adjacent angle of the polygon.

41°. Given two spherical arcs together less than a semicircumference, to place them so that, with a third not given, they may contain the greatest surface possible.

42°. Through a given point to describe a great circle which shall touch two given equal and parallel small circles.

43°. To inscribe a circle in a given spherical triangle.

SECTION SECOND.

Projections of the Sphere.

PROPOSITION I.

The Orthographic Projection of every circle of the sphere, as meridians and parallels of latitude, will be an ellipse, circle, or straight line, according as its plane shall be oblique, parallel, or perpendicular to the plane of projection. (573)

1°. For let M, N, P, Q, be the segments of any two chords, $M+N$, $P+Q$, of a circle, and m, n, p, q, their orthographic projections upon the plane of $m+n$, $p+q$; that is, projections made by perpendiculars let fall from the extremities of M, N; P, Q, we have,

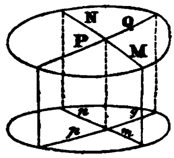

Fig. 122.

$$\left.\begin{aligned} M \cdot N &= P \cdot Q, \\ m = M\cos(M,m),\ n &= N\cos(M,m); \\ p = P\cos(P,p),\ q &= Q\cos(P,p); \end{aligned}\right\} \quad (574)$$

$$\therefore \qquad \frac{mn}{\cos^2(M,m)} = \frac{pq}{\cos^2(P,p)} \qquad (575)$$

which determines the nature of the curve of projection.

2°. If $M+N$ pass through the centre, O, and $P+Q$ be at right angles to $M+N$, then P will $= Q$, and $\therefore$ (574) p will $= q$; whence it follows that $m+n$ bisects a system of parallel chords $p+q$, and is itself bisected in o, the projection of O; therefore o is the centre of the curve. Assuming o as the origin of a system of oblique coordinates, (575) becomes,

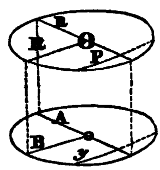

Fig. 122₂.

$$\frac{(A+x)(A-x)}{\cos^2(M,m)} = \frac{yy}{\cos^2(P,p)},$$

putting $m+n = 2A$; but R being the radius of the circle and B the projection of that R which is parallel to P, we find

$$A = R\cos(M,m),$$
$$B + R\cos(P,p);$$

$$\therefore \qquad \frac{R^2}{A^2}(A^2 - x^2) = \frac{R^2}{B^2}\cdot y^2,$$

or
$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1. \tag{576}$$

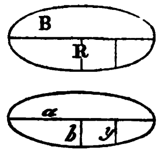

Fig. 122₃.

3°. If R be assumed in such position that its projection, A, or, for the sake of distinction, a, shall be parallel to it and consequently b, the new value of B, perpendicular to a, there results

$$\left.\begin{aligned} a &= R\cos(M,m) = R\cos 0^\circ = R, \\ b &= R\cos(P,p) = R\cos I, \end{aligned}\right\} \tag{577}$$

I being the inclination of the circle to the plane of projection; we have also

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{578}$$

which is the equation of an ellipse of which the axes are $2a$, $2b$; and the proposition is demonstrated, since when $I = 0$, b becomes parallel to and equal to $R = a$, and when $I = 90^\circ$, b becoming $= 0$, the ellipse vanishes in a straight line.

Cor. 1. The ellipse may be referred to a system of oblique coördinates such that its equation (576) shall be of the same form with that obtained for rectangular axes (578), and $2A$, $2B$, mutually bisecting each other and all chords drawn parallel to them, are denominated *Conjugate Diameters*, of which the axes $2a$, $2b$, are but particular values. (579)

Cor. 2. If two systems of parallel chords intersect each other in an ellipse, the products of their segments will be proportional (575); and this property may be extended to the case in which the points of intersection lie without the curve. (580)

Cor. 3. An elliptical arc, MmApP, being given, the centre, O, may be found by bisecting AOB drawn through the middle points of any parallel chords, MP, mp. (581)

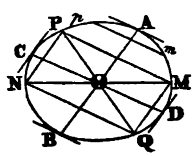

Fig. 122₄.

Cor. 4. A conjugate, CD, to any diameter, AB, may be found by drawing CD through the middle points of AB and PN a chord parallel to AB. (582)

Cor. 5. A parallelogram, MPNQ, may be described in (583) an ellipse by drawing chords, MQ, PN; MP, QN, parallel to the conjugates AB, CD.

Cor. 6. The diagonals MN, PQ, mutually bisecting each (584) other in O, are obviously diameters, and P may be so chosen that MN shall be the conjugate axis $= 2a$, in which case PM, PN, are said to be *Supplementary Chords*.

Cor. 7. The tangents drawn through the extremities (585) A, B, C, D, of the conjugate diameters, are parallel to the supplementary chords, and to the corresponding diameters; and, by their intersections, constitute a parallelogram circumscribing the ellipse.

PROPOSITION II.

To make an Orthographic Projection of the Sphere.

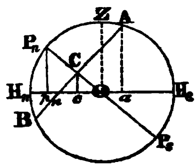
Fig. 123.

With the radius OA, equal to that of the required map, describe the meridian AZP_nH_nB P_sH_s, passing through the north and south poles, P_n, P_s, and, consequently, perpendicular to the horizon seen edgewise in H_nH_s. Take the arc H_nP_n measuring the elevation of the pole, equal to the latitude of the place Z over which the observer is supposed to be situated at a very great [infinite] distance. Having graduated the meridian and drawn the parallels of latitude ACB, which will be perpendicular to the axis P_nOP_s, drop their extremities and centres, A, C, in the projections a, c, also project the elevated pole P_n in p_n. Transfer the points a, c, p_n, to the central meridian H_nOH_s of the map, and through c draw a perpendicular to H_nH_s, intersecting the circumference in m, m_2; cm and ca will be the semimajor and semiminor axes of the elliptical parallel of latitude mam_2, which may be described by the first exercise under the ellipse. Next, for the meridians, let $H_1P_nH_n = h$ be the angle which any required meridian, P_nH_1 makes with the vertical meridian P_nH_n, and $H_nH_1 = H$, the corresponding arc intercepted on the horizon, further P_nH_n will $= l$, the latitude of the place upon whose horizon the projection is made; therefore, by Napier's Rules, $P_nH_nH_1$ being a right angle, we have

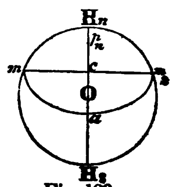
Fig. 123a.

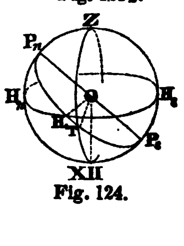
Fig. 124.

$$\cos l_c = \cot h \cot H_c,$$

or

$$\tan H = \sin l \tan h. \qquad (586)$$

Having laid off $H_n H_1 = H$ found by calculation, and drawn the diameter $H_1 O$, and Ox perpendicular to it, transfer the semimajor axis $H_1 O$ to the straight edge of a thin ruler or slip of paper, and, having applied this line in $p_n x$, and marked its point, y, of intersection with OH_1, proceed to describe the meridian $H_1 p_n$ according to the first exercise under the ellipse.*

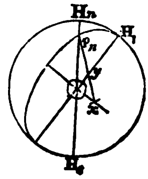

Fig. 124a.

The work now described is to be combined in a single figure, when the features of the spherical surface will be laid down according to the projections of the meridians and parallels of latitude.

Fig. 125.

PROPOSITION III.

To make the Gnomonic Projection when the dial is horizontal.

Let $H_n H_1 H_s$ be the horizontal face of the dial, O its (*fig.* 124) centre and OP_n the style elevated according to the latitude of the place; the shadow at noon will fall upon the north point, H_n, and its positions, OH_1, for all other hours will be determined by (586), where h = the hour angle.

PROPOSITION IV.

To make the Gnomonic Projection of the Style upon a vertical south plane.

The construction will be similar to the preceding, (*fig.* 124) observing that,

$$\tan(\mathrm{XII})T = \cos l \tan h \qquad (587)$$

* An instrument very convenient for this purpose may be constructed, consisting of a slender ruler carrying a pen or pencil in its extremity and furnished with two moveable pins, one to glide in a groove cut in the stem of a wooden T and the other along its top.

PROPOSITION V.

If an oblique circular cone be truncated by a plane at right angles to that plane which passes through the axis and is perpendicular to the base, the section will, in general, be an ellipse. (588)

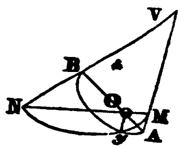

Fig. 126.

Let AoOBy be the truncating plane perpendicular to the plane, AMVBN, passing through the axis at right angles to the base; draw $oy = y$ perpendicular to AB, and pass the plane, MNy, parallel to the base, it will be a circle, and the common ordinate, $oy = y$, will be perpendicular to the diameter, MoN;

$$\therefore \qquad (oy)(oy) = (o\text{M})(o\text{N}).$$

But
$$\frac{o\text{M}}{o\text{A}} = \frac{\sin o\text{AM}}{\sin o\text{MA}}, \text{ and } \frac{o\text{N}}{o\text{B}} = \frac{\sin o\text{BN}}{\sin o\text{NB}};$$

$\therefore$, putting OA = OB = a, Oo = x, and $b = y_{x=0}$,

$$y^2 = \frac{\sin o\text{AM}}{\sin o\text{MA}} \cdot \frac{\sin o\text{BN}}{\sin o\text{NB}} (a^2 - x^2);$$

and
$$b^2 = \frac{\sin o\text{AM}}{\sin o\text{MA}} \cdot \frac{\sin o\text{BN}}{\sin o\text{NB}} \cdot a^2;$$

$$\therefore \qquad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}, \text{ the equation of an ellipse.}$$

Cor. The section becomes a circle when the angles, which the truncating plane and circular base make with the sides of the cone, are equal and contrary. (589)

For, when the angle oAM = oNB, then $\angle$ oBN = oMA, and the equation becomes

$$y^2 = a^2 - x^2, \text{ that of a circle.}$$

PROPOSITION VI.

The Stereographic Projection of any circle of the sphere is, in general, itself a circle. (590)

In the stereographic projection every point of the spherical surface is thrown upon an equatorial plane, denominated the *primitive* circle, by the intersection with this plane of a line drawn through the point and the eye situated in the pole of the primitive.

Now, let AB be that diameter of any circle,

whose extremities lie in the circumference, $AP_nBOH_tP_tEH_n$, passing through the poles, E, O, of the primitive $H_nH_1H_tH_2$, whose centre, o, is the same with the centre of the sphere, and draw AE, BE, intersecting H_nH_t in a, b; ab is the stereographic projection of AB. We have,

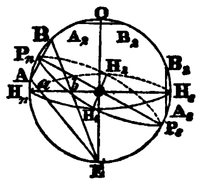

Fig. 127.

$$\text{measure of } \angle EBA = \tfrac{1}{2}\text{arc}EH_nA = \tfrac{1}{2}(EH_t + H_nA) = \text{meas. of } \angle Eab;$$

$$\therefore \quad \angle EBA = Eab,$$

and (589) the projection of the circle described on AB is itself a circle situated in the primitive $H_nH_1H_tH_2$, and having the diameter, ab.

PROPOSITION VII

The distance from the centre of the primitive at which any point will be projected, will be equal to the tangent of half the arc intercepted between the point and the pole of the primitive, the radius of the sphere being taken for unity. (591)

For let A be any point on the surface of the sphere ; we have

$$\frac{oa}{oE} = \frac{\sin oEa}{\sin oaE} = \tan oEa,$$

or $$oa = \tan\tfrac{1}{2}OA, \ [oE = 1]. \quad (591)$$

Cor. 1. If m be the centre of the circle, ab, we have,

$$\text{dist. of centre } om = \tfrac{1}{2}(oa + ob) = \tfrac{1}{2}(\tan\tfrac{1}{2}OA + \tan\tfrac{1}{2}OB), \quad (592)$$

where it is to be observed that oa, ob, or their equivalents, $\tan\frac{1}{2}OA$, $\tan\frac{1}{2}OB$, change signs on A, B, passing O.

Cor. 2. The radius $ma = \frac{1}{2}(\tan\frac{1}{2}OA - \tan\frac{1}{2}OB)$. (593)

Cor. 3. The projection of a circle parallel to the primitive will be concentric with the primitive. (594)

For, if A_2B_2 be parallel to H_nH_t, we have

$$OA_2 = OB_2, \quad \therefore \quad om_2 = \tfrac{1}{2}(\tan\tfrac{1}{2}OA_2 - \tan\tfrac{1}{2}OA_2) = 0,$$
$$m_2a_2 = \tfrac{1}{2}(\tan\tfrac{1}{2}OA_2 + \tan\tfrac{1}{2}OA_2) = \tan\tfrac{1}{2}OA_2.$$

Cor. 4. If the pole of the circle be in the primitive, the distance of its projected centre will be the secant of its radius, and the radius of projection the tangent of the same arc. (595)

For, if we put $H_tA_3 = H_tB_3 = u$,

then $$\tfrac{1}{2}OA_3 = 45^\circ + \tfrac{1}{2}u, \ \tfrac{1}{2}OB_3 = 45^\circ - \tfrac{1}{2}u;$$

and $\therefore$ (592), (593), reduced by (356), become

$$om_3 = \frac{1}{\cos u} = \sec u,$$

$$m_3 a_3 = \frac{\sin u}{\cos u} = \tan u.$$

Cor. 5. A great circle perpendicular to the primitive will be projected in a diameter of the primitive. (596)

For we have (595),

$$om_{u-90°} = \sec 90° = \textit{infinity},$$
$$ma_{u-90°} = \tan 90° = \textit{infinity}.$$

The same is obvious from the figure.

Cor. 6. Any great circle will have for the distance of its projected centre, the tangent of the arc measuring its inclination to the primitive, and the radius of projection will be the secant of the same arc. (597)

Let P_nP_s be the diameter of any great circle, $P_nH_1P_sH_2$, and imitate the reasoning in (595.)

Scholium I. Any point situated in the primitive, is its own projection. (598)

Scholium II. If a great circle and the primitive intersect each other in the diameter, H_1H_2, and a third circle passing through the axis of the primitive in the diameters, P_nP_s, H_nH_s, the points, H_1, H_2, may be found by spherical trigonometry, and the point, p_n, the projection of P_n, having been determined by (591), there will be three points given through which to pass the circumference, $H_1p_nH_2$, the projection of $H_1P_nH_2$. (599)

H_nH_1 will be found (586),

if we put $\quad H_nH_1 = H$, $H_nP_nH_1 = h$, $H_nP_n = l$.

PROPOSITION VIII.

To make the Stereographic Projection of the Sphere.

I. Let the eye be situated at the south pole in order to project the northern hemisphere. Describe the primitive and graduate its circumference according to the number of meridians which it is intended to lay down; these will be diameters passing through the several points of division (596).

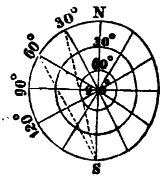

Fig. 123.

For the parallels of latitude, we have only to count off from the

north point the corresponding degrees, and from the points of division to draw lines to the south point, and the intersections thus formed with the east and west diameter will be in the circumferences of the circles of latitude whose centres will be that of the primitive, (591), (594).

II. Let the eye be in the equator. The primitive will pass through the poles, N, S, and the central meridian, NS, and equator, EQ, will be north and south, and east and west lines intersecting in the centre of the primitive, (596). Graduate the circumference as above, and, laying the corner of a wooden square upon a point of division, direct the edge of one arm through the centre. The intersection of the corresponding edge of the other arm with the production of the line, NS, will be the centre, and the distance of this point to the point of division the radius, for the description of a parallel of latitude, (595). The radii just employed set off from the centre upon EQ will give the centres for describing the meridians, which will pass through the points, N, S, (597), (598).

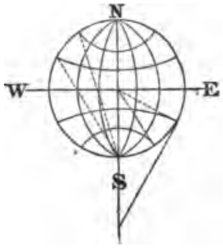

Fig. 129.

III. When the eye is situated otherwise than as above, consult (598), (599), and figure 130.

Scholium. It will be observed that the middle of the map is comparatively contracted in the stereographic and enlarged in the orthographic projection. To avoid this, the lines, NS, EQ, are sometimes divided into equal parts, and the map thus constructed is improperly denominated a *globular* projection.

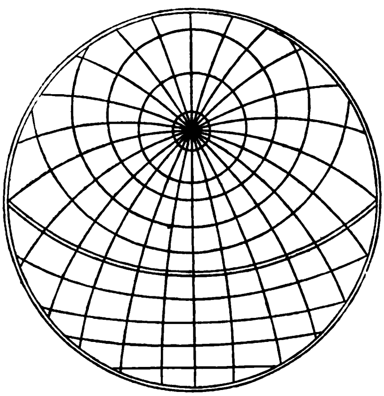

Fig. 130.

PROPOSITION IX.

To make the Conical Projection.

In this projection, the eye, situated at the centre of the sphere, throws every point of its surface upon that of a tangent or secant cone, the axis of which is coincident with the axis of the sphere.

Let a sphere and cone be generated by the revolution of the arc, NnsES, and the straight line, vns, intersecting each other in n, s, about the common axis, vN$o_n o_s$OS, O being the centre of the sphere, N and S the poles, E a point of the equator, and no_n, so_s, perpendiculars upon NS.

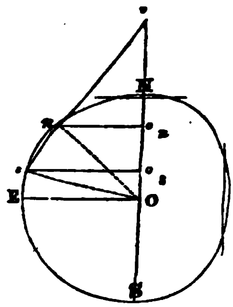

Fig. 131.

1°. For the magnitude of a parallel of latitude, putting

OE $=$ Os $=$ On $=$ ON $=$ OS $=$ radius $= 1$,

arcA°_n $=$ length of an arc of A° in the parallel through n,

arcA°_s $=$do. through s, arcA°_E $=$ do. through E;

l_n $=$ latitude of n,

l_s $=$ latitude of s,

we have,

$$\left.\begin{array}{ll} & no_n = \sin nN = \cos l_n, \\ & so_s = \sin sN = \cos l_s; \\ \therefore & \text{arc}A^\circ_n : A^\circ_s = \cos l_n : \cos l_s, \\ \text{and} & \text{arc}A^\circ_n : A^\circ_E = \cos l_n : 1, \\ & \text{arc}A^\circ_s : A^\circ_E = \cos l_s : 1. \end{array}\right\} \quad (600)$$

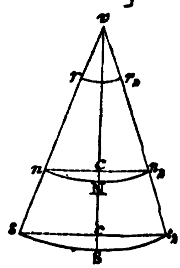

Fig. 131₂

2°. For the radii, vn, vs, with which to describe the sectoral surface, snN$n_2 s_2$S, the development of the conical surface upon a plane, we have,

$$\begin{array}{ll} & vn : no_n = \sin 90^\circ : \sin nvo_n, \\ \text{or} & vn : \cos l_n = 1 : \sin\tfrac{1}{2}(s\text{S} - n\text{N}); \end{array}$$

$$\left.\begin{array}{ll} \therefore & \text{radius } vn = \dfrac{\cos l_n}{\sin\frac{1}{2}(l_n + l_s)}, \\ \text{and} & \text{radius } vs = \dfrac{\cos l_s}{\sin\frac{1}{2}(l_n + l_s)}. \end{array}\right\} \quad (601)$$

3°. To find the number of degrees in the angle, svs_2, corresponding to a difference of latitude of A°; with the trigonometrical radius, $vr = 1$, describing the arc, rr_2, there results,

$$\text{arc}rr_2 \;:\; \text{radius } vr = \text{arc}nn_2 \;:\; \text{radius } vn,$$

or $$\text{arc}v^\circ \;:\; 1 = \text{arc}A^\circ_n \;:\; \frac{\cos l_n}{\sin(\frac{1}{2}l_n + l_s)};$$

$$\therefore \quad \text{arc}v^\circ = \text{arc}A^\circ_E \cdot \sin\tfrac{1}{2}(l_n + l_s). \qquad (602)$$

Now, to make the plane of projection approximate the spherical surface as much as may be conveniently done, take n and s so that these and the middle point of the map shall divide its meridian equally, or as nearly so as the degrees of latitude or their aliquot parts will permit. Calculate the radii, vn, vs (601), and find the degrees, v° (602), corresponding to the degrees of longitude, A°, to be embraced by the map; from a scale of the required magnitude, lay off the meridian, $\text{NS} = vs - vn$, and produce it in v, about which as a centre, and with the radii, vn, vs, taken from the same scale, describe the arcs, $n\text{N}n_2$, $s\text{S}s_2$, and, having made the arc, $rr_2 = v^\circ$, and drawn vns, vn_2s_2, divide the arc, $n\text{N}n_2$, into A parts; through these points of division and corresponding ones in NS and NS produced, draw the remaining meridians and parallels of latitude.

4° To find the half chords, cn, cs, and their altitudes, $\text{N}c$, $\text{S}c$.

$$nc \;:\; nv = \sin\tfrac{1}{2}v \;:\; \sin 90^\circ,$$

$$\left.\begin{aligned} &\therefore \quad \text{half chord } cn = \sin\tfrac{1}{2}v \cdot \frac{\cos l_n}{\sin\frac{1}{2}(l_n + l_s)}, \\ &\text{and} \quad cs \;:\; cn = \cos l_s \;:\; \cos l_n. \end{aligned}\right\} (603)$$

Again, $$vc \;:\; vn = \cos\tfrac{1}{2}v \;:\; 1;$$

$$\therefore \quad vc = \cos\tfrac{1}{2}v \cdot \frac{\cos l_n}{\sin\frac{1}{2}(l_n + l_s)};$$

$$\therefore \quad \text{N}c = (1 - \cos\tfrac{1}{2}v) \cdot \frac{\cos l_n}{\sin\frac{1}{2}(l_n + l_s)},$$

$$\left.\begin{aligned} &\text{or} \quad \text{N}c = 2\sin^2\tfrac{1}{4}v \cdot \frac{\cos l_n}{\sin\frac{1}{2}(l_n + l_s)}; \\ &\text{so} \quad \text{S}c = 2\sin^2\tfrac{1}{4}v \cdot \frac{\cos l_s}{\sin\frac{1}{2}(l_n + l_s)}. \end{aligned}\right\} (604)$$

Finally, $$\text{NS} = \frac{\cos l_s - \cos l_n}{\sin\frac{1}{2}(l_n + l_s)} = 2\sin\tfrac{1}{2}(l_n - l_s). \qquad (605)$$

Forms (603), (604), (605), enable us to make the construction with accuracy when the point, v, is so distant as not to admit of employing the radius, vn, by finding the successive values of v° (602) for $A = 1^\circ, 2^\circ, 3^\circ, \ldots$.

Scholium I. This projection is well adapted to the construction

of maps of moderate extent from north to south; and when a greater number of degrees of latitude are to be embraced, it will also become quite accurate by regarding the conical surface as composed of several placed end to end, joining each other in parallels little distant, as 1°, 2°, or 30′.

Scholium II. When the points, n, s, are situated at equal distances on opposite sides of the equator, E, the cone becomes a cylinder; for we have,

$$l_s = -l_n \quad \therefore \quad (601), \text{ radius } vn = \frac{\cos l_n}{\sin 0°} = \text{infinity},$$

and (605), $\text{NS} = 2\sin l_n$.

Scholium III. When the points, n, s, are taken on opposite sides of the pole, N, and at equal and moderate distances, the projection becomes a circular plane, well calculated to represent the circumpolar regions, either of the earth or the heavens.

EXERCISES.

1°. Construct a map of the sphere as it would appear to an observer elevated to the zenith of lat. 43° N., lon. 78° W.

2°. Project the sphere as, supposing it transparent, its opposite surface would appear on the rational horizon to an eye situated in the surface of the place just given.

3°. Make a conical projection of the State of New York and the countries extending a degree or two on the north and south, assuming the centre of the map near Syracuse, and calculating the radii to each degree of latitude.

4°. Prove that the section made by a plane through the vertex of a right circular cone, is an isosceles triangle, passing into a straight line by a revolution of the plane.

5°. Show that if a plane cut a right circular cone parallel to its side, the section will be a parabola, vanishing in a straight line.

6°. The truncation of a right circular cone will be found to be an ellipse, passing on the one side into a circle and a point, and on the other into a parabola.

7°. If a right circular cone be cut by a plane not passing through the vertex, not truncating it, nor parallel to its side, the section will be a hyperbola, whose limits will be a straight line or parabola.

BOOK THIRD.

NAVIGATION.

SECTION FIRST.

Problem of the Course.

The earth is an *oblate spheroid*, formed by the revolution of an ellipse about its minor axis, the polar diameter being 7899·170 and the equatorial 7925·648 English miles; but, for the purposes of Navigation, it may be regarded as a sphere with a radius of 3437·75 geographical miles or minutes.

If a ship could be conveniently guided in the arc of a great circle, which would be the most direct path, nothing would be easier than, by Spherical Trigonometry, to determine her course from one point on the earth's surface to another, or, having given the point of departure and the distance run, to find her place. But as, in practice, the vessel is directed by the magnetic needle so as to cut all the meridians at a constant angle, the path actually described is a curve of double curvature, denominated the *Rhumb Line*, or *Loxodromic Curve*.

PROPOSITION I

To find the difference of latitude.

Let a ship sail from X_1 to X and describe the rhumb z, intersecting the meridians x_1, x, $x+h$, at the constant angle c = *course*. P is the pole, x_1 the polar distance of X_1, x that of X, and $y = \angle(x_1, x)$ = the difference of longitude; k, h, i, are the vanishing increments of y, x, z, and j is a corresponding vanishing arc described about

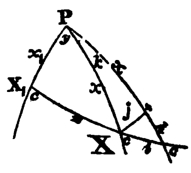

Fig. 132.

P with the radius, x, which may therefore be regarded as the arc of a great circle perpendicular to $x + h$.

It is required to find the function $x = \phi z$.

Assuming the earth's radius for unity, we have

$$x'_{x=\phi z} = \left[\frac{h}{i}\right] = \frac{\sin(90^\circ - c)}{\sin 90^\circ} = \cos c$$

$\therefore \quad x = z\cos c + constant,$

and $\therefore \quad x_1 = x_{z=0} = constant;$

$\therefore \quad x - x_1 = z \cos c, \qquad (606)$

or Difference of Latitude = Distance $\times$ cosine of Course.

PROPOSITION II.

To find the difference of Longitude.

Let $y = \psi x$ be the function required; there results,

$$y'_{y=\psi x} = \left[\frac{k}{h}\right] = \left[\frac{k}{j}\right] \cdot \left[\frac{j}{h}\right] = \left[\frac{\sin k}{\sin j}\right] \cdot \left[\frac{j}{h}\right]$$

$$= \frac{\sin 90^\circ}{\sin x} \cdot \frac{\sin c}{\sin(90^\circ - c)},$$

or $\quad y'_{y=\psi x} = \tan c \cdot \frac{1}{\sin x} = \tan c \cdot \frac{\sin x}{(\sin x)^2} = \frac{\sin x}{1 - \cos^2 x} \cdot \tan c,$

but [[(298)]], $\quad \left[l\left(\frac{1 + \cos x}{1 - \cos x}\right)^{-\frac{1}{2}}\right]' = \frac{-(\cos x)'}{1 - \cos^2 x} = \frac{\sin x}{1 - \cos^2 x};$

$\therefore \quad y = \left[l\left(\frac{1 + \cos x}{1 - \cos x}\right)^{-\frac{1}{2}}\right] \cdot \tan c + constant,$

but, (351) $\quad \left(\frac{1 + \cos x}{1 - \cos x}\right)^{-\frac{1}{2}} = \left(\frac{1 - \cos x}{1 + \cos x}\right)^{\frac{1}{2}} = \left(\frac{(1 - \cos x)^2}{1 - \cos^2 x}\right)^{\frac{1}{2}}$

$$= \frac{1 - \cos x}{\sin x} = \tan\tfrac{1}{2}x;$$

$\therefore \quad y = \tan c \cdot l \tan\tfrac{1}{2}x + constant,$

and $\therefore \quad 0 = \tan c \cdot l \tan\tfrac{1}{2}x_1 + constant;$

$\therefore \quad y = \tan c \cdot l \frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}x_1}$

But, in order to adapt this form to the purposes of computation, we must convert the Napierian into common logarithms, and adopt, as is customary, for the unit of measure the nautical mile or minute instead of the earth's radius, which will be done by multiplying by $\frac{1}{M} = 2{\cdot}30258509$ and $3437{\cdot}75$; and there results,

$$\text{dif. lon. } y = 7915{\cdot}705 \tan c \cdot \log. \frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}x_1}. \qquad (607)$$

$$[\log. 7915{\cdot}705 = 3{\cdot}8984896].$$

Cor. 1. If X_1 be a point of the equator, or $x_1 = 90°$,

then, $\quad y = 7915{\cdot}705 \tan c \log. \tan(45° + \frac{1}{2}l)$, (608)

putting $\quad$ latitude $(x - 90°) = l$.

Cor. 2. If we imagine the spherical surface rolled out upon a plane so that the equator, y, shall become a straight line, the parallels of latitude and their distances apart being increased for the purpose, the distance, z_2, the difference of longitude, y, reckoned on the equator, and the increased difference of latitude, m, estimated from the equator, will form the three sides of a right angled triangle, and we find

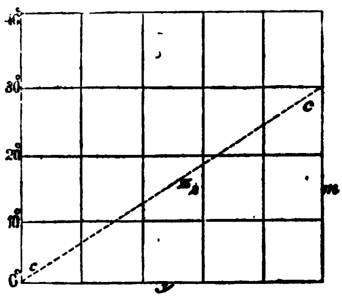

Fig. 133.

$$\tan c = \frac{y}{m}, \text{ or } y = m \tan c;$$

$$\therefore (608), \quad m = 7915{\cdot}705 \log. \tan(45° + \tfrac{1}{2}l). \qquad (609)$$

When a table is constructed by aid of (609), called *Meridional Parts*, the problems in navigation relating to the course may be solved by the rules of plane trigonometry; and a *Chart* constructed from these numbers, bears the name of Mercator.

PROPOSITION III.

To find the difference of longitude when the ship sails on a parallel of latitude.

In this case cos$c$ becoming $= 0$, form (606) will be inapplicable; but since z is now a parallel of latitude and $x = x_1$, $=$ a constant, i coinciding with j, we find,

$$y'_{y-Fz} = \left[\frac{k}{i}\right] = \left[\frac{k}{j}\right] = \frac{\sin 90°}{\sin x_1} = \frac{1}{\sin x_1};$$

$$\therefore \qquad y = \frac{1}{\sin x_1} \cdot z, \qquad (610)$$

which accords with (600).

Scholium. There are various methods of solving the Problem of the Course.

1°. *Plane Sailing*, where the distance run being small, the sea is regarded as a plane, and the common forms of plane trigonometry are employed.

2°. *Traverse Sailing*, where several short courses are run in succession, and forms (421), (422), are applicable.

3°. *Parallel Sailing*, where the ship sails on a parallel of latitude, and the difference of longitude will be found by (610).

4°. *Middle Latitude Sailing*, where the distance run is moderate and nearly east and west; in this method the *middle* latitude is assumed to be equal to the *half sum* of the two extreme latitudes when both are of the same name.

5°. *Mercator's Sailing*, as explained in (609).

EXERCISES.

1°. A ship sails from 3° S. lat. and 9° W. lon., N. 40° W. 538 miles. Required her latitude and longitude.

For the difference of latitude we have (606),

$$\log.\cos 40^\circ = \bar{1}{\cdot}884254$$
$$\log. 538 = 2{\cdot}730782$$
$$\therefore \qquad \text{dif. lat.} = 412{\cdot}13',\ 2{\cdot}615036$$

For the difference of longitude (607),

$$3{\cdot}89849$$
$$\log.\tan 40^\circ = \bar{1}{\cdot}92381$$
$$\left.\begin{array}{l}\log.\tan\tfrac{1}{2}x = 0{\cdot}02950\\ \log.\tan\tfrac{1}{2}x_1 = \bar{1}{\cdot}97725\end{array}\right[\text{South Pole.}]$$
$$\log.\frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}x_1} = 0{\cdot}05225 \qquad \log.\log.\frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}x_1} = \bar{2}{\cdot}71809$$
$$2{\cdot}54039$$

dif. long. = 347 miles.

2°. In what direction must a ship sail from New York to arrive at London, and what distance will she make?

3°. A ship sails from 20° N. L. and 70° W. L. for the Azores in 38° N. L. and 30° W. L. After performing a thousand miles of her course, she tacks for New York; in what direction must

she steer, and what will be her distance to the last-mentioned port?

4°. If a ship could sail from the equator N. 45° E. till she should arrive at the pole, what would be her distance run, and what her progress in longitude?

The distance run will be found = 7636'68 miles, a finite number, but the difference of longitude will be infinite; that is, the ship will make an infinite number of turns about the pole, spinning there at length, like a top, with an infinite velocity of rotation.

5°. Calculate the following table of meridional parts, and employ it in solving the exercises just given, and in constructing a chart extending from the equator 60° each way.

Table of Meridional Parts.

°		10°	20°	30°	40°	50°	60°	70°	80°
0°	0	603	1225	1888	2623	3474	4527	5966	8375
1°	60	664	1289	1958	2702	3569	4649	6146	8739
2°	120	725	1354	2028	2782	3665	4775	6335	9145
3°	180	787	1419	2100	2863	3764	4905	6534	9606
4°	240	848	1484	2171	2946	3865	5039	6746	
5°	300	910	1550	2244	3030	3968	5179	6970	
6°	361	973	1616	2318	3116	4074	5324	7210	
7°	421	1035	1684	2393	3203	4183	5474	7467	
8°	482	1098	1751	2468	3292	4294	5631	7745	
9°	542	1161	1819	2545	3382	4409	5795	8046	

SECTION SECOND.

Problem of the Place.

Owing to the impossibility of sailing absolutely in the required course and measuring accurately the distance run, it becomes necessary to resort to astronomical observations for the purpose of determining from time to time the ship's position on the ocean. To this end the daily places of the most conspicuous of the heavenly

bodies are computed and laid down in the Nautical Almanac* for Greenwich *mean* time, and for that of Paris in the Connaissances des Tems.† These places are such as they would appear to an observer at the earth's centre, and are estimated in right ascension and declination.

The *declination* of a heavenly body is its angular distance from the *Equinoctial* [*Celestial Equator*], plus if north, minus when south, and the distance measured, commonly in hours, from the *Vernal Equinox*, to the east upon the *Equinoctial*, is the *right ascension* of the same body.

PROPOSITION I.

Given the geocentric meridian altitude of a heavenly body and its declination, to find the latitude.

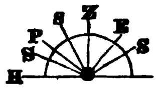

Fig. 134.

Let O be the centre of the earth, P the pole elevated above the horizon, H, and 90° distant from the celestial equator, E; then denoting the zenith by Z and the position of any heavenly body in the same meridian by S, S, S, we have

$$ZOE = HOP = HOS \pm POS, \text{ or,}$$
$$L = P_h = S_h \pm S_p, \text{ i. e.,}$$

The latitude of a place, or the altitude of the elevated pole, is equal to a heavenly body's meridian distance from the point of the horizon under the pole, increased or diminished by the polar distance of the same body, according as it is situated below or above the pole. (611)

Scholium I. The polar distance of any heavenly body is equal to 90° $\mp$ its declination. (612)

Scholium II. The latitude may be found on shore, or when the vessel is stationary, by observing that the elevation of the pole (613)

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* Blunt's reprint is sufficient for all purposes except those of a fixed and regularly furnished observatory, and is much less expensive. The student should be possessed of such astronomical knowledge and acquaintance with the stars as may be acquired from elementary works like "Kendall's Uranography."

† To make use of either ephemeris it will evidently be necessary to reduce the time of any observation to that of the meridian for which the ephemeris is calculated, by allowing 15° to the hour, and observing the equation of time.

= the half sum of the upper and lower meridian altitudes of any circumpolar star; since

$$P_h = (S_h)_u - S_p \text{ and } P_h = (S_h)_l + S_p$$

give $$P = \tfrac{1}{2}[(S_h)_u + (S_h)_l].$$

Example. On the 20th of August, 1847, at noon, the meridian altitude of the sun's centre was found to be 62° 49′ 50″; at the same instant the chronometer, regulated to Greenwich time indicated 2h. 45m. Required the latitude of the place.

Interpolating (279) for the declination from the Nautical Almanac, we have the following

Operation.

Day of Month.	Declination.		
20	$y_0 = 12° 35' 19''.0$	$-19' 49''.2 = D_1$	
21	12 15 29.8	20 00.8	$+11''.6 = D_2$
22	11 55 29.0		11.4 [∴ $D_3 = 0$]
23	11 35 16.8	20 12.2	

$$x = 2\text{h. }45\text{m.} = \frac{2 + \frac{45}{60}}{24} = \tfrac{11}{96}\text{d.}$$

$$\begin{array}{rl} D_1 = & -19' \; 49''.2 \\ \cdot\, 11 & \;\; 19\;492 \\ \hline 96)209 & 541.2 \\ \hline xD_1 = & -2 \;\; 16.3 \end{array} \qquad \begin{array}{rl} \tfrac{1}{2}D_2 = & +\; 5''.3 \\ & \;\; 53 \\ \hline & 96)58.3 \;[\cdot\, \tfrac{11}{96}] \\ \hline & \;\; .6 \\ \hline & -\,.5 \;[\cdot\, (\tfrac{11}{96} - 1)] \end{array}$$

$$\begin{array}{lrl} \therefore & y_0 = & 12° \; 35' \; 19''.0 \\ & xD_1 = & \;\; -2 \; 16.3 \\ & x(x-1)\cdot\dfrac{D_2}{2} = & \;\;\;\; -.5 \\ \hline \therefore & \text{N. D.} = y_x = & 12 \; 33 \;\; 2.2 \\ \hline \therefore & \text{N. P. Dist.} = & 77 \; 26 \; 57.8, \; [-], \; (612) \\ \hline & \text{S's alt.} = & 62 \; 49 \; 50.0 \\ \hline \therefore & S_h = & 117 \; 10 \; 10.0 \\ \hline \therefore & & 39° \; 43' \; 12''.2 \text{ Lat. N., required.} \end{array}$$

PROPOSITION II

Given the polar and zenith distances of a heavenly body, and the zenith distance of the pole [= colatitude], to find the time.

Fig. 135.

The hour, found approximately by the longitude of the vessel and the chronometer, regulated to Greenwich time, is to be corrected by observation.

Let A be the pole, B the position of the heavenly body, and C the zenith of the place; then the hour angle A will be found by (540) or by (541), the first being preferable when A is small. (614)

Example. In the preceding example, after sailing 3 hours due west, the zenith distance of the sun's centre was found to be $47^\circ\ 43'\ 51{\cdot}3''$. Required the hour angle, A, or the time referred to the meridian of the place arrived at.

Operation.

$$3\text{h.} = \tfrac{1}{8}\text{d.} \qquad 8)\,-19'\ 49{\cdot}2'' \qquad 8)\,+5{\cdot}3''$$

$$\underline{2\ 28{\cdot}65} \qquad \begin{array}{r} {\cdot}66 \\ 8 \\ \hline -{\cdot}58\ [{\cdot}\ (\tfrac{1}{8}-1)]; \end{array}$$

$$\therefore \quad \begin{array}{rrrr} & {}^\circ & {}' & {}'' \\ & 12 & 33 & 2{\cdot}2 \\ & & -2 & 28{\cdot}65 \\ & & & -{\cdot}58 \\ \hline \text{N.D.} = & 12 & 30 & 33; \\ \hline\hline a = & 47 & 43 & 51{\cdot}3 \\ b = & 50 & 16 & 47{\cdot}8 \\ c = & 77 & 29 & 27{\cdot}0 \\ \hline & 2)175 & 30 & 6{\cdot}1 \\ \hline h = & 87 & 45 & 3{\cdot}05 \\ \hline\hline \end{array}$$

$$\begin{array}{rrrrl}
 & {}^\circ & ' & '' & \\
h-b = & 37 & 28 & 15\cdot25 & \bar{1}\cdot78\ 41\ 595 \\
h-c = & 10 & 15 & 36\cdot05 & \bar{1}\cdot25\ 07\ 018 \\
\hline
b = & 50 & 16 & 47\cdot8 & \bar{1}\cdot88\ 60\ 257 \\
c = & 77 & 29 & 27\cdot0 & \bar{1}\cdot98\ 95\ 661 \\
\cline{5-5}
 & & & & 2)\bar{1}\cdot15\ 92\ 695 \\
\cline{5-5}
 & & & & \bar{1}\cdot57\ 96\ 347
\end{array}$$

$$\therefore \qquad \tfrac{1}{2}A = 22^\circ\ 19'\ 32''\cdot2$$

$$\text{and} \qquad A = 44\ \ 39\ \ 4\cdot4$$

or $\qquad A = 2$h. 58m. 36·3s., time required.

Therefore the ship had sailed 20·92$\frac{2}{3}$ miles.

Scholium I. The time of setting of a heavenly body, or, rather, the hour angle when the body is 90° distant from the zenith, will be better found by (569), since we have

$$\cos A_{,} = \cot b \cot c,$$

or
$$\cos(180 - A) = \tan \text{lat.} \times \tan \text{dec.}, \qquad (615)$$

where it is to be remembered that the declination will become minus when measured in a direction opposite to the elevated pole.

Ex. When did the sun set to the place in the last example but one? *Ans.* 6h. 42m. 38s.

— 19s. for change of dec.

Scholium II. The time of the apparent setting of a heavenly body will be found by substituting for a, the apparent zenith distance of the horizon, which will generally differ from 90°. (616)

Scholium III. The hour when twilight ends will be found by putting $a = 90° + 18° = 108°$, since the sun is 18° below the horizon at that time. (617)

Ex. When did twilight end in the above example?

Ans. 7h. 3m. 27s.

Scholium IV. The azimuth, C, or bearing of a heavenly body, when its zenith distance, polar distance, and either the hour angle or the zenith distance of the pole, are known, will be found by (551), or by (540). (618)

Ex. What was the bearing of the sun at setting in the above?

Ans. 73° 40′ 14″.

How might the variation of the magnetic needle be determined?

Scholium V. When on shore, the magnetic variation will be determined, with greater accuracy, by ascertaining the greatest east-

ern or western elongation of a circumpolar star, *i. e.*, its azimuth when $B = 90°$. We have (568)

$$\cos c_c = \sin b \sin C, \text{ or } \log.\sin C = \log.\sin c - \log.\sin b. \qquad (619)$$

For this purpose the pole star is preferable. Its polar distance on the 1st of January, 1847, was 1° 30′ 22·85″, and decreases 19·273″ annually.

The time of greatest elongation will be found by taking the difference of the right ascensions of the sun and star from the Nautical Almanac. The elongation will be easily observed by the aid of any altitude and azimuth instrument, carefully levelled, as a theodolite—or, if no such instrument be at hand, by suspending a long plumb line, with its weight swimming in a pail of water to prevent agitation by the wind, and, at a suitable distance south of the line, placing upon a table, a piece of board carrying a sight vane, by the motion of which, east or west, the required point will be obtained when the star appears no longer to depart from the line. Next alline a lighted candle at a considerable distance north—ten or a dozen rods—which is to be blown out and left till morning, when it may be observed with a common compass, or the true meridian may be traced by calculating a triangle, and constructing it with a rod graduated for measuring lengths. In making the above observation it will be necessary to illumine the plumb-line in the direction of the star.

Scholium VI. When the zenith distance, polar distance, and either the hour angle or the azimuth, of a heavenly body are given, the latitude may be found by combining (535) with (383 ... 4); for we obtain

$$\tan z = \frac{\cos c}{\sin c \cos A}, \text{ and } \sin(b + z) = \frac{\cos a \cos z}{\sin c \cos A}, \qquad (620)$$

or

$$\tan z_2 = \frac{\cos a}{\sin a \cos C}, \text{ and } \sin(b + z_2) = \frac{\cos c \cos z_2}{\sin a \cos C}. \qquad (621)$$

Having determined by the aid of a theodolite, on the same or different evenings, the altitudes and azimuths of the same or different stars, the mean of several computations by (621) will give the latitude of a place on land with a considerable degree of accuracy. The same observations will also make known the hour

The error that is likely to be committed in the hour angle, *A*, the chronometer not indicating the exact time, may be corrected by trial, from a couple of observations, the first made soon after

the meridian passage and the other several hours later. It is to be observed that a change of sign in the error of A will correspond to a change of sign in the error of b, and that A is to be varied till two values of b are found answering to the difference of latitude made by the vessel between the observations.

Scholium VII. THE METHOD OF DOUBLE ALTITUDES has for its object to determine the latitude and time from two zenith distances, either of the same body taken a few hours apart, or of different bodies taken at the same time.

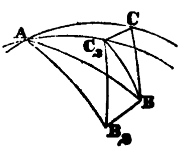


Fig. 135 a.

Let CB be the first zenith distance of the sun, and C_2 the zenith arrived at when the second C_2B_2 is taken; the number of miles the ship sails gives the number of minutes in the arc CC_2, and the angle, $BCC_2 = C$, is determined by the course and the azimuth of B at the first observation.

Therefore, in the triangle, BCC_2, to find c, we have

$$\cos c = \cos b \cos c_2 + \sin b \sin c_2 \cos C;$$

but, as b is small and c differs little from c_2, putting $c = c_2 - d$, and expanding by (317), reserving only the first and second powers of d and b, the reduced equation becomes,

$$d^2 - 2 \tan c_2 \cdot d = b^2 - 2b \tan c_2 \cos C, \qquad (622)$$

or, better, $\quad c_2 - c = d = b \cos C, \qquad (623)$

when, as in the present case, the second powers of the small quantities, d, b, may be rejected.

There are now three steps in the operation: (624)

1°. In the triangle, BAB_2, we have the two sides, AB, AB_2, and the included angle A = the time elapsed between the observations, or to the difference of right ascensions, if two stars are observed at the same time, to find the remaining parts;

2°. Therefore the angle, C_2B_2B, becomes known from the three sides of the triangle, C_2B_2B;

3°. And, lastly, in the triangle AB_2C_2, we have the two sides, B_2A, B_2C_2, and the included angle, B_2, to find the colatitude, AC_2, and the hour angle, C_2AB_2.

Note 1. When the declination changes but little between (625) the observations, the first part of the operation may be shortened by regarding the triangle, BAB_2, as isosceles, taking instead of AB or AB_2, $\frac{1}{2}(AB + AB_2)$ for the equal side and solving by (570).

Note 2. In any of the preceding operations, when a star, (626)

instead of the sun, is employed, the hour angle will be converted into degrees by observing that a sidereal day is equal to 23h. 56m. 4·0906s.; whence it follows that

$$1 \text{ solar hour} = 1·002738 \text{ sidereal h.} = 15°·04107.$$

Note 3. Before taking out any quantity from the Nautical Almanac, or the Connaissances des Tems, the time of the place where the observation was made must be reduced to the meridian and denomination of time specified in the ephemeris. "Equations" (as these numbers are technically called) for reducing apparent to mean time, will be found in the ephemeris itself. (627)

Example. On the first day of August, 1847, at 10 o'clock 43m. 25s. P. M., apparent time, and in 20° W lon., it is required to find the moon's declination.

Ap. time of pla.		Diff. lon.		Ap. time at Gr.
10h. 43m. 25s.	+	$1\frac{1}{3}$h.	=	11h. 3m. 25s.

Ap. time at Gr.		Eq. time.		Mean time at Gr.
11h. 3m. 25s.	+	6m. 1 s.	=	11h. 9m. 26s.

$\therefore$ 6° 27′ 13·6″ + 1′ 40·8″ = 6° 28′ 54·4″ N.Dec. *Ans.*

Example. At the time specified in the example preceding the last, the ship tacks and sails N 43° W, nine and a half miles an hour for two hours, when the zenith distance of the sun's centre is found to be 37° 55′ 40″. Required the latitude of the place arrived at and the exact time.

PROPOSITION III.

The Lunar Method for the Longitude consists in finding by the ephemeris the time corresponding to the geocentric distance of the moon from a heavenly body. The problem requires for its solution the apparent angle which these bodies make with each other, as well as their apparent and geocentric zenith distances. (628)

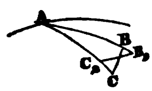


Fig. 136.

Let A, B, C, be the places of the zenith, moon, and some other heavenly body, as the sun, as seen from the earth's centre. On account of *parallax* and *refraction*, to be explained hereafter, the moon will appear at B_2, below B and in the same vertical, and the second body will be seen in C_2, vertically above C; hence B_2C_2 will be the apparent distance of the two luminaries.

There are now three steps in the operation for the longitude: (629)

1°. In the triangle B_2AC_2 we have the three sides a_2, b_2, c_2, the apparent distances of the two luminaries and the zenith, to find their difference of azimuth, A_1.

2°. In the triangle BAC, we have the two sides, b, c, and the included angle A, to find the third side, a, or the geocentric distance of the moon from the heavenly body.

3°. Lastly, with a, we enter the table of Lunar Distances and take out, by interpolation, the corresponding Greenwich time, the difference between which and that of the place, converted into degrees, will be the longitude required.

Scholium. The lunar distance may be calculated from the right ascensions and declinations of the two heavenly bodies; since the codeclinations and the difference of right ascensions constitute two sides and the included angle of a spherical triangle of which the third side is the distance required. (630)

Example. On the 9th April, 1837, at 5h. 29m. 36·8s. mean time, the apparent distance of the centres of the sun and moon was found to be $B_2C_2 = a_2 = 55^\circ\ 1'\ 18{\cdot}8''$; at the same instant the apparent zenith distance of the moon was $AB_2 = c_2 = 28^\circ\ 49'\ 50''$ the true $AB = c = 28^\circ\ 27'\ 48''$, and the corresponding data for the sun were $AC_2 = b_2 = 68^\circ\ 9'\ 46''$, $AC = b = 68^\circ\ 12'\ 10''$. The moon was east of the sun and both west of the meridian; the estimated latitude was 41° 47′ N, and the longitude 2h. 10m. west, which gives, approximately, 7h. 40m. for the Paris time.

True distance $BC = a = 55^\circ\ 17'\ 20''$[calculated].

We find in the *Conn. des Tems*, that

at 6h. Paris time	$a = 54^\circ\ 11'\ 36''$,
difference	$1^\circ\ 5'\ 44''$;

the change in a for 3h. was 1° 25′ 55″,

∴ 1° 25′ 55″ : 1° 5′ 44″ : : 3h. :	2h. 17m. 42·9s.,
the corresponding hour at Paris was	8h. 17m. 42·9s.,
and the exact hour of the place was	5h. 29m. 36·8s. ;
∴ the required longitude was	2h. 48m. 6·1s. W.

N. B. This result may be corrected by commencing the calculation anew with 2h. 48m. instead of 2h. 10m.

SECTION THIRD.

Description and Use of Instruments.

Besides the Chronometer already alluded to, the principal instruments of the navigator are the *Compass*, the *Log*, and the *Sextant.*

PROPOSITION I.

To steer a determinate course.

This is effected by aid of the *Mariner's Compass*, which consists, essentially, of a circular card, poised on a pivot and carrying a magnetic needle. The circumference of the card is divided into thirty-two equal parts, denominated *points*, and each point into *quarters.* The bearings, called *rhumbs*, are estimated from the north and south toward the east and west, as follows ;

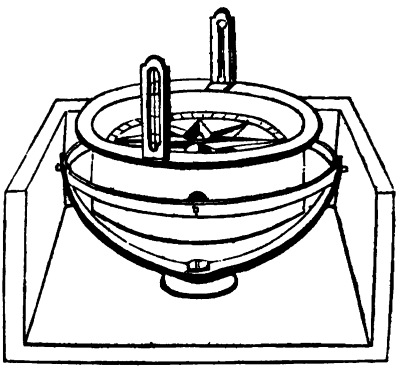


Fig. 136.

N, N by E, NNE, NE by N, NE, NE by E, ENE, E by N, E;
N, N by W, NNW, NW by N, NW, NW by W, NNW, W by N, W;
S, S by E, SSE, SE by S, SE, SE by E, ESE, E by S, E;
S, S by W, SSW, SW by S, SW, SW by W, WSW, W by S, W.

It is easy, therefore, to convert any rumb into degrees;

thus, $\quad$ SE by E = S 5 points E = S $\frac{5}{8}\cdot 90^\circ$ E = S $56^\circ\ 15'$ E.

PROPOSITION II.

To determine the Ship's Rate.

This is accomplished by a line, divided at one extremity into two branches which

Fig. 136b.

are attached to the vertical radius of a wooden sector, having its arc so loaded with a strip of lead as to swim buried in the sea with its face perpendicular to the line of draft. The instrument is denominated the *Log and Line.* The *log-line* is divided into equal parts, called *knots*, of which 120 = a nautical mile; hence the number of knots run off from a light reel while a half minute glass is emptying, corresponds to the miles which the ship sails in an hour.

PROPOSITION III.

To determine the zenith distance of a heavenly body.

I. *The apparent altitude above the apparent horizon* is observed with the sextant. This instrument, so useful to the surveyor and astronomer and indispensable to the navigator since it may be held in the hand without any fixed support, consists essentially of an arc embracing 60° and graduated as 120°, of an index, I, carrying upon the centre, O, a mirror, MON, of a second glass, PGQ, one half silvered, and of a telescope, T. Besides, there are screws for fastening and tilting the mirror, MN, for tilting and turning G, for tilting and elevating or depressing the telescope, T, and for fastening the index, I, and moving it over a small arc when once clamped.

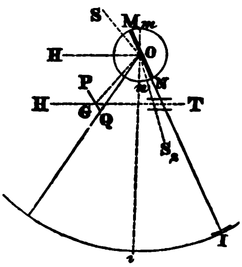


Fig. 137.

Fig. 137a.

Now let the index, I, take up such a position that the moveable mirror, MN, shall be parallel to the fixed, PQ, and let a ray of light, HO, fall upon MN in a direction to be reflected in OG and again from G through the telescope, T, to the eye. Since, by a principle in Optics, the incident and reflected rays make equal angles with the mirrors, we have

$$\angle \text{HOM} = \text{GON}$$

and $$\text{OGP} = \text{TGQ};$$

$$\therefore \quad \angle \text{HOM} = \text{TGQ} = \text{GTM};$$

hence, the rays HO and HGT being parallel, the two images of the distant object from which they proceed, the one seen by reflection from G and the other through the clear part, will appear to coincide in the direction TGH. But a ray, SO, falling in any other direction, being reflected in OS_2, will not enter the telescope, and, in order to make it do so, we advance the index, I, till arriving at i, OS_2 coincides with OG. There results,

$$\angle n\text{OG} = m\text{OS} = m\text{OM} + \text{MOS},$$

$$n\text{OS}_2 = n\text{ON} - \text{NOS}_2 = m\text{OM} - \text{MOS};$$

$$\therefore \quad \text{GOS}_2 = 2m\text{OM},$$

but $$\text{HOS} = \text{HOM} - \text{MOS} = \text{GON} - \text{NOS}_2 = \text{GOS}_2;$$

$$\therefore \quad \angle \text{HOS} = 2m\text{OM} = 2\text{arc } \text{I}i, \text{ i. e.,}$$

The angle measured by bringing into coincidence the images of two objects, one seen by reflection and the other directly, is double the arc passed over by the index. (631)

The reasons for the following adjustments will now be obvious. (632)

First Adjustment. To make the *index glass*, O, perpendicular to the plane of the instrument. Bring the index to the middle of the graduated arc, and, looking into the mirror, observe whether the limb seen directly and by reflection appear broken or continuous.

Second Adjustment. To make the *horizon glass*, G, perpendicular to the plane of the instrument. Having interposed the colored glasses, direct the telescope to the sun, or, better at night, to a star of the first magnitude, when, on moving the index backward and forward, the two images, in passing each other, should exactly coincide.

Third Adjustment. To make the *line of collimation*, or the axis of the telescope, parallel to the plane of the instrument. Turn the eye-piece till the two wires stretched parallel to each other in the common focus of the lenses for the purpose of restricting the observer to the centre of the field of view, become parallel to the plane of the instrument; next, bring two objects, as the edges of the sun and moon, distant by 90° or more, into coincidence upon one of the wires; the images should continue to coincide when, by turning the instrument a little, they are made to appear on the sec-

ond wire. The above adjustments should be repeated one after the other till all are quite accurate.

Fourth Adjustment. To determine the *index error*, or to make the zeros of the vernier and limb agree. The error will be determined by bringing the reflected and direct images of a star or planet into coincidence, or, by observing the readings when the images of the sun are brought into contact on opposite sides.

Fifth Adjustment. To make the reflected and direct images equally bright. Turn the *up and down piece* which carries the telescope to and from the plane of the instrument, till the object is accomplished.

For a description of the *Reflecting Circle*, see Francœur's "Géodésie."

Now, for example, suppose it be required to find the apparent altitude of the sun's lower limb above the apparent horizon. Turn the shades which are situated in front of the mirrors into the lines OG, HG, so as properly to soften both the reflected and direct rays; next, set the index at zero and direct the telescope to the sun; push the index a little till the lower image is out of sight, following the upper one; drop the screen in front of the horizon glass that the sky may appear, and continue the motion of the index till the sun's image touches the horizon; clamp, and balance the instrument gently about the line TGH, alternately producing and breaking contact, the better to see when the tangency becomes exact, and this will be accomplished by carefully turning the tangent screw, which impresses a small motion upon the index. A microscope may be employed for reading the angle, and one is sometimes attached to the index. (633)

Scholium I. On shore an artificial horizon may be employed, consisting of a basin of mercury, from the surface of which the light of the celestial body is reflected. The measured distance, SMS_2, of the two images will be double the apparent altitude, SMH. (634

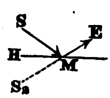


Fig. 138.

Scholium II. As a convenience in finding the time, we may choose a point of observation, and determine, once for all, the hour angle of a terrestrial object, sufficiently remote and distinct, from which to measure the angular distance of the sun, whenever required. (635)

Scholium III. In obtaining the data for the longitude, three arcs, measured at the same instant, are required; viz., the (636)

altitudes of the moon and a star or the sun and the lunar distance. The three measures may, however, be executed by a single observer, by taking at equal and short intervals of time, 1°, the altitude of the sun or star; 2°, the altitude of the moon; 3°, the lunar distance; 4°, a second altitude of the sun; 5°, a second altitude of the moon—from which the altitudes simultaneous with the lunar distance will be readily found.

II. *For the depression of the horizon.* Let A be the position of the observer, elevated above the level of the sea by the altitude h, let T be the apparent horizon, O the earth's centre and r its radius, also let $\angle$ AOT $= e$, and the depression TAH [= AOT] $= d$. We find

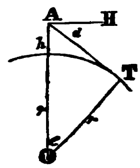


Fig. 139.

$$\tan d = \frac{\sin e}{\cos e} = \frac{AT}{OT} = \frac{\sqrt{(2rh + h^2)}}{r},$$

which may be reduced to

$$d = \sqrt{\frac{2}{r}} \cdot \sqrt{h}, \text{ or } d'' = \frac{648000}{\pi} \cdot \sqrt{\frac{2}{r}} \cdot \sqrt{h}, \qquad (637)$$

observing that h is small compared with r, and, consequently, $\tan d = d$ nearly, also that an arc of $1''$, $\left(= \frac{\pi}{180 \cdot 60 \cdot 60}\right)$ is contained in the radius (= 1), $\frac{648000}{\pi}$ times.

Now, from the aggregate of many observations made on shore, with an instrument adjusted by a spirit level, there results,

h	d	r	$\log. \left(\frac{648000}{\pi} \cdot \sqrt{\frac{2}{r}}\right)$
Observed.	Observed.	Calculated.	Calculated.
10 feet.	186'76″	24396000 feet.	1'77128
20 "	264'12	24395000 "	1'77128
30 "	323'48	24395000 "	1'77128.

For calculating the depression then we have

$$\log. d'' = 1{\cdot}77128 + \tfrac{1}{2}\log. h. \qquad (638)$$

Scholium. It will be observed that the value 24395000 (639) is greater than the true radius, which, according to Dr. Bowditch, may be taken at 20911790 feet. We infer, therefore, that a ray of light, passing tangent to the sea, is bent into a curve concave toward the earth's centre; since (637), if the true value of r be taken,

h will be smaller than the value assumed in the figure, and more and more so the greater d becomes, h varying as the square of d; consequently the path of the ray will fall below the tangent, TA, and continually depart from it.

III. *For the refraction.* We will suppose an observer provided with an astronomical clock, carefully regulated to sidereal time (24h. = 360°), and an altitude and azimuth instrument, accurately leveled so that the upper and lower meridian transits of a circumpolar star shall take place at intervals of twelve hours as exactly as possible, and small, unavoidable errors allowed for. The data, both observed and calculated,* requisite for determining the law of atmospheric refraction, may be tabulated as follows.

[Last of Aug., 1847.]

Ap. zen. dist. of polaris. at upper transit = 45° 33′ 50″,
do. do. do. at lower transit = 48° 34′ 10″
∴ ap. zen. dist. of pole [calculated] = 47° 4′.

Note. The fixed stars are so distant that they suffer no displacement on account of parallax, appearing in the same direction whether seen from the earth's centre, or from any point on its surface.

Now, the azimuths and hour angles of a Cygni being observed at the successive zenith distances, 15°, 30°, 45°, 60°, 75°, 80°, 85°, 90°, in the triangle ABC, we have (*fig.* 135.)

A = observed hour angle, a_2 = ob. zen. dis., a = calcul. zen. dis.
b = cal. zen. dis. of pole = 47° 4′, C_2 = ob. azimuth, C = cal. az.
c = polar dis. of star = 45° 15′ 41·5″, [Naut. Al.]

Barometer 30 *inches, internal Thermometer* 50°, *external* 47°.

A	a_2	a	C_2	C	$a - a_2$	$(a - a_2) : \tan a_2$
1h. 22m. 49·81s.	15°	15° 00′ 15·5″			00′ 15·5″	57·847′
2 47 59·16	30	30 00 33·6			00 33·6	58·200
4 16 15·10	45	45 00 58·1			00 58·1	58·100
5 51 7·40	60	60 01 40·5			00 40·5	58·024
7 40 53·62	75	75 03 34·3			03 34·3	57·421
8 24 53·48	80	80 05 20·0			05 20·0	56·425
9 17 48·33	85	65 09 58·0			09 58·0	52·318
10 40 45·56	90	90 33 51·0			33 51·0	00·000

The values of C_2 are found the same with those of C, which, therefore, the student may calculate and verify by employing them to find the values of A.

* The student should find the calculated numbers for himself.

It appears from the above that the effect of the atmospheric refraction is to elevate a heavenly body, without changing its azimuth, by an angle which varies, to within 10° or 15° of the horizon, nearly as the tangent of the zenith distance. With this limitation, therefore, we may assume

$$(a - a_2) : \tan a_2 = 58''$$

$$\therefore \quad a = a_2 + 58'' \tan a_2 \qquad (640)$$

But the refraction is found to vary with the density of the atmosphere, at 45° zenith distance increasing about 2″ for every additional inch of the barometer $[B]$ above 30 inches $[B - 30]$, and decreasing ·12″ for each additional degree of Fahrenheit's thermometer over 50° $[T - 50]$; hence (640) becomes

$$a = a_2 + [58 + 2(B - 30) - \cdot 12(T - 50)]'' \tan a_2. \qquad (641)$$

Scholium. On account of the fluctuating nature of the atmosphere, observations within 15° or 20° of the horizon should be avoided. For the variation of the magnetic needle, however, the azimuth of the sun may be taken when the lower limb appears above the horizon by $\frac{1}{3}$ its diameter.

IV. *For the parallax.* Let S be the position of a heavenly body seen from a point, A, of the earth's surface, O the centre, Z the zenith, and S_h the position of the same body in the horizon;* let $\angle$ ASO $= p$, ZAS $= a$, ZOS $= a_c$; $AS_hO = p_h$, $ZAS_h = 90°$; then p, the difference of a and a_c, is called the *parallax in altitude*, p_h the *horizontal parallax*, and we have, putting OS $= d$, OA $= r$,

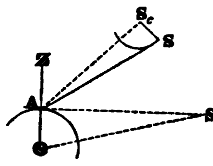


Fig. 140.

$$\sin p : \sin a = r : d, \text{ and } \sin p_h : \sin 90° = r : d;$$

$$\therefore \quad \sin p = \sin p_h \sin a, \text{ or } p = p_h \sin a, \qquad (642)$$

when the parallaxes are small.

$$\therefore \quad a_c = a - p_h \sin a. \qquad (643)$$

Scholium. The horizontal parallaxes are given in the Nautical Almanac.

V. *For the semidiameter.* Let S be the point of contact of the line SA and a heavenly body, as the sun, whose centre denote by S_c; there results,

$$\sin SAS_c : \sin AS_cS = SS_c : AS\left[= \frac{d \sin a_c}{\sin a}\right],$$

* These positions are supposed to be corrected for refraction.

or $\quad \tan \text{semidi. } s = \tan SAS_c = \frac{SS_c}{d} \cdot \frac{\sin a}{\sin a_c};$

but, if we denote by s_h what s becomes when the point S falls a little below S_h so as to make $\angle SAO = SOA$ (in which case it is obvious that s_h may be taken for the horizontal semidiameter), we have

$$\tan s_h = \frac{SS_c}{d}; \qquad [\sin a = \sin a_c.]$$

$$\therefore \qquad \tan s = \tan s_h \cdot \frac{\sin a}{\sin a_c},$$

or semidiameter $s = \text{hor. sem. } s_h \cdot \frac{\sin a}{\sin a_c}$, nearly. (644)

In (643), replacing $a_c = ZOS$, and a by ZOS_c and $a - s$, we find

$$ZOS_c = a - s - p_h \sin(a - s), = a_c - s, \text{ nearly.} \qquad (645)$$

Scholium I. The refraction for the ray S_cA is less than that for SA; therefore, denoting this difference by dif. ref. (S_c,S), when great accuracy is required, as in finding the longitude, (645) becomes

True zen. dist. $ZOS_c = a - s - p_h \sin(a - s) + \text{dif. ref. } (S_c,S)$. (646)

Scholium II. Except for the moon, we may assume (647) $s = s_h$.

Example. On the 20th of August, 1847, at 6h. Greenwich mean time, the eye of the observer being 24 feet above the level of the sea, the altitude of the moon's lower limb above the apparent horizon was found to be 40° 20′. At the same time the barometer indicated 29 inches, and the thermometer 40°, Required the true geocentric zenith distance of the moon's centre.

	40° 20′ 00·0″
d (638)	4′ 49·3″
Ap. alt. from tr. horizon,	40° 15′ 10·7″
Ap. zen. dis. of moon's lower limb,	49° 44′ 49·3″
Refraction (641)	1′ 7·6″
True zen. dis., a, of moon's lower limb,	49° 45′ 56·9″
Parallax, p, (642), [p_h = 56′ 24·5″ N. A.]	43′ 3·7″
Tr. geocentric zen. dis. of moon's lr. limb a_c,	49° 2′ 53·2″
Semidiameter, s, (644) [s_h = 15′ 22·3″ N. A.]	15′ 35·0″
True geocentric zen. dis. of moon's centre	48° 47′ 16·2″

The student may make further corrections (646).

ADDENDUM I.

The ingenious theorem contained in the following communication is published by permission. G. C. W.

RICHMOND, *June 22d*, 1848.

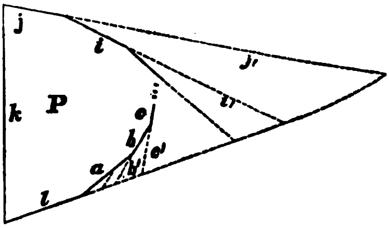
PROFESSOR WHITLOCK,

Dear Sir:—I have modified the demonstration of my theorem as you requested, and hasten to send you the result.

I shall, at the earliest opportunity, attend to the problem which you suggested, to find the area of a polygon in terms of its sides and angles, excepting two sides and one angle. If I arrive at anything of importance, I will send it to you.

Yours respectfully,
DASCUM GREEN.

To find a general expression for the area of a polygon.



Let $(a, b, c, \ldots, j, k, l,)$ be any polygon. Produce the sides $b, c, \ldots, i, j$, till they meet l, or l produced, and denote the corresponding productions by $b', c', \ldots, j'$.

Since the expression for the double area of a triangle (a, b, c), in terms of its three angles and one side is

$$\frac{a^2 \sin(a,b) \sin(a,c)}{\sin(b,c)},$$

we shall have, if we designate the area of the polygon $a, b, \ldots, h$ by P,

$$2P = \frac{(j+j')^2 \sin(k,j) \sin(j,l)}{\sin(l,k)} - \frac{(i+i')^2 \sin(j,i) \sin(i,l)}{\sin(j,l)} - \dots$$

$$- \frac{(b+b')^2 \sin(c,b) \sin(b,l)}{\sin(c,l)} - \frac{a^2 \sin(b,a) \sin(a,l)}{\sin(b,l)}$$

$$= \frac{a^2 \sin(a,b) \sin(a,l)}{\sin(b,l)} + \frac{(b+b')^2 \sin(b,c) \sin(b,l)}{\sin(c,l)}$$

$$+ \frac{(c+c')^2 \sin(c,d) \sin(c,l)}{\sin(d,l)} +$$

$$+ \frac{(j+j')^2 \sin(j,k) \sin(j,l)}{\sin(k,l)},$$

in which it remains to determine b', c', ..., j'.

We have $\dfrac{b'}{a} = \dfrac{\sin(a,l)}{\sin(b,l)}$, $\therefore\ b' = \dfrac{a \sin(a,l)}{\sin(b,l)}$;

$$\frac{c'}{b+b'} = \frac{\sin(b,l)}{\sin(c,l)},\ \therefore\ c' = \frac{(b+b') \sin(b,l)}{\sin(c,l)}$$

$$= \frac{a \sin(a,l) + b \sin(b,l)}{\sin(c,l)};\ \text{\&c., \&c., \&c.};$$

$$\frac{j'}{i+i'} = \frac{\sin(i,l)}{\sin(j,l)},$$

$$\therefore\ j' = \frac{(i+i') \sin(i,l)}{\sin(j,l)} = \frac{a \sin(a,l) + b \sin(b,l) + \dots + i \sin(i,l)}{\sin(j,l)}.$$

Substituting these values, we obtain

$$2P = \frac{a^2 \sin(a,b) \sin(a,l)}{\sin(b,l)} + \frac{\left[b + \dfrac{a \sin(a,l)}{\sin(b,l)}\right]^2 \sin(b,c) \sin(b,l)}{\sin(c,l)}$$

$$+ \frac{\left[c + \dfrac{a \sin(a,l) + b \sin(b,l)}{\sin(c,l)}\right]^2 \sin(c,d) \sin(c,l)}{\sin(d,l)} + \dots$$

$$+ \frac{\left[j + \dfrac{a \sin(a,l) + b \sin(b,l) + \dots + i \sin(i,l)}{\sin(j,l)}\right]^2 \sin(j,k) \sin(j,l)}{\sin(k,l)};$$

$$\therefore\ 2P = [a \sin(a,l)]^2 \cdot \frac{\sin(a,b)}{\sin(a,l) \sin(b,l)}$$

$$+ [a \sin(a,l) + b \sin(b,l)]^2 \cdot \frac{\sin(b,c)}{\sin(b,l) \sin(c,l)}$$

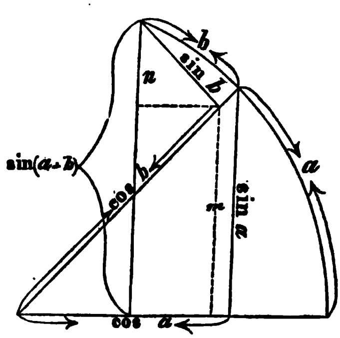
$$+ a\sin(a,l) + b\sin(b,l) + c\sin(c,l)]^2 \cdot \frac{\sin(c,d)}{\sin(c,l)\sin(d,l)} + \ldots$$

$$+ [a\sin(a,l) + b\sin(b,l) + \ldots + j\sin(j,l)]^2 \cdot \frac{\sin(j,k)}{\sin(j,l)\sin(k,l)},$$

which is the expression required, and which embraces the sides $a, b, c, \ldots, j$, and the angles $(a,l), (b,l), (c,l), \ldots, (k,l)$; $(a,b), (b,c), (c,d), \ldots, (j,k)$.

ADDENDUM II.

The student who may desire a partial course in geometry and trigonometry, may omit all after page 108 to p. 170, excepting pp. 145, 153, 154, 155, 156, 157, 158, 159, which are to be read, and substitute for Propositions V, VI, VII, VIII, on pages 172, 173, 174, 175, the following demonstration, after which pages 176, 177, 178, 191, 192, 193, 194, 195, 196, 197, 198, 199, 200 will be read, and then the Book on surveying.



Let a, b, be two adjacent arcs of the same circumference, and $a+b < 90°$; draw $\sin a$ intercepting $\cos a$, $\sin b$ intercepting $\cos b$, $\sin(a+b)=m+n$, m being a perpendicular from the foot of $\sin b$ upon $\cos a$, and n the portion of $\sin(a+b)$ above a parallel to $\cos a$ drawn through the foot of $\sin b$; r being the radius of the circle, from similar triangles we have the proportions,

$$\frac{m}{\cos b}=\frac{\sin a}{r},\ \therefore\ m=\frac{\sin a \cos b}{r},$$

$$\frac{n}{\sin b}=\frac{\cos a}{r},\ \therefore\ n=\frac{\cos a \sin b}{r};$$

$$\therefore\ \sin(a+b)=m+n=\frac{\sin a \cos b+\cos a \sin b}{r},$$

or, when $r=1$, $\sin(a+b)=\sin a \cos b+\cos a \sin b$;

$\therefore$ $\sin[a+(-b)]=\sin a \cos(-b)+\cos a \sin(-b)$,

or (180), (306), $\sin(a-b)=\sin a \cos b-\cos a \sin b$;

$\therefore$ (302), $\cos(a+b)=\sin[90°-(a+b)]$

$$=\sin[(90°-a)-b]$$

$$=\sin(90°-a)\cos b-\cos(90°-a)\sin b$$

$$=\cos a \cos b-\sin a \sin b;$$

$\therefore$ (180), (306), $\cos(a-b)=\cos a \cos b+\sin a \sin b$,

and (320) is demonstrated.

LOGARITHMS OF NUMBERS.

from 1 to 1000

$$y_x = y_0 + x \cdot \frac{D_1}{1} + x(x-1) \cdot \frac{D_2}{1 \cdot 2} + x(x-1)(x-2) \cdot \frac{D_3}{1 \cdot 2 \cdot 3}.$$

	100	110	120	130	140	150	160
0	0000000	0413927	0791812	1139434	1461280	1760913	2041200
1	0043214	0453230	0827854	1172713	1492191	1789769	2068259
2	0086002	0492180	0863598	1205739	1522883	1818436	2095150
3	0128372	0530784	0899051	1238516	1553360	1846914	2121876
4	0170333	0569049	0934217	1271048	1583625	1875207	2148438
5	0211893	0606978	0969100	1303338	1613680	1903317	2174839
6	0253059	0644580	1003705	1335389	1643529	1931246	2201081
7	0293838	0681859	1038037	1367206	1673173	1958997	2227165
8	0334238	0718820	1072100	1398791	1702617	1986571	2253093
9	0374265	0755470	1105897	1430148	1731863	2013971	2278867
	170	180	190	200	210	220	230
0	2304489	2552725	2787536	3010300	3222193	3424227	3617278
1	2329961	2576786	2810334	3031961	3242825	3443923	3636120
2	2355284	2600714	2833012	3053514	3263359	3463530	3654880
3	2380461	2624511	2855573	3074960	3283796	3483049	3673559
4	2405492	2648178	2878017	3096302	3304138	3502480	3692159
5	2430380	2671717	2900346	3117539	3324385	3521825	3710679
6	2455127	2695129	2922561	3138672	3344538	3541084	3729120
7	2479733	2718416	2944662	3159703	3364597	3560259	3747483
8	2504200	2741578	2966652	3180633	3384565	3579348	3765770
9	2528530	2764618	2988531	3201463	3404441	3598355	3783979
	240	250	260	270	280	290	300
0	3802112	3979400	4149733	4313638	4471580	4623980	4771213
1	3820170	3996737	4166405	4329693	4487063	4638930	4785665
2	3838154	4014005	4183013	4345689	4502491	4653829	4800069
3	3856063	4031205	4199557	4361626	4517864	4668676	4814426
4	3873898	4048337	4216039	4377506	4533183	4683473	4828736
5	3891661	4065402	4232459	4393327	4548449	4698220	4842998
6	3909351	4082400	4248816	4409091	4563660	4712917	4857214
7	3926970	4099331	4265113	4424798	4578819	4727564	4871384
8	3944517	4116197	4281348	4440448	4593925	4742163	4885507
9	3961993	4132998	4297523	4456042	4608978	4756712	4899585

	310	320	330	340	350	360	370
0	4913617	5051500	5185139	5314789	5440680	5563025	5682017
1	4927604	5065050	5198280	5327544	5453071	5575072	5693739
2	4941546	5078559	5211381	5340261	5465427	5587086	5705429
3	4955443	5092025	5224442	5352941	5477747	5599066	5717088
4	4969296	5105450	5237465	5365584	5490033	5611014	5728716
5	4983106	5118834	5250448	5378191	5502284	5622929	5740313
6	4996871	5132176	5263393	5390761	5514500	5634811	5751878
7	5010593	5145478	5276299	5403295	5526682	5646661	5763414
8	5024271	5158738	5289167	5415792	5538830	5658478	5774918
9	5037907	5171959	5301997	5428254	5550944	5670264	5786392
	380	390	400	410	420	430	440
0	5797836	5910646	6020600	6127839	6232493	6334685	6434527
1	5809250	5921768	6031444	6138418	6242821	6344773	6444386
2	5820634	5932861	6042261	6148972	6253125	6354837	6454223
3	5831988	5943926	6053050	6159501	6263404	6364879	6464037
4	5843312	5954962	6063814	6170003	6273659	6374897	6473830
5	5854607	5965971	6074550	6180481	6283889	6384893	6483600
6	5865873	5976952	6085260	6190933	6294096	6394865	6493349
7	5877110	5987905	6095944	6201361	6304279	6404814	6503075
8	5888317	5998831	6106602	6211763	6314438	6414741	6512780
9	5899496	6009729	6117233	6222140	6324573	6424645	6522463
	450	460	470	480	490	500	510
0	6532125	6627578	6720979	6812412	6901961	6989700	7075702
1	6541765	6637009	6730209	6821451	6910815	6998377	7084209
2	6551384	6646420	6739420	6830470	6919651	7007037	7092700
3	6560982	6655810	6748611	6839471	6928469	7015680	7101174
4	6570559	6665180	6757783	6848454	6937269	7024305	7109631
5	6580114	6674530	6766936	6857417	6946052	7032914	7118072
6	6589648	6683859	6776070	6866363	6954817	7041505	7126497
7	6599162	6693169	6785184	6875290	6963564	7050080	7134905
8	6608655	6702459	6794279	6884198	6972293	7058637	7143298
9	6618127	6711728	6803355	6893089	6981005	7067178	7151674
	520	530	540	550	560	570	580
0	7160033	7242759	7323938	7403627	7481880	7558749	7634280
1	7168377	7250945	7331973	7411516	7489629	7566361	7641761
2	7176705	7259116	7339993	7419391	7497363	7573960	7649230
3	7185017	7267272	7347998	7427251	7505084	7581546	7656686
4	7193313	7275413	7355989	7435098	7512791	7589119	7664128
5	7201593	7283538	7363965	7442930	7520484	7596678	7671559
6	7209857	7291648	7371926	7450748	7528164	7604225	7678976
7	7218106	7299743	7379873	7458552	7535831	7611758	7686381
8	7226339	7307823	7387806	7466342	7543483	7619278	7693773
9	7234557	7315888	7395723	7474118	7551123	7626786	7701153
	590	600	610	620	630	640	650
0	7708520	7781513	7853298	7923917	7993405	8061800	8129134
1	7715875	7788745	7860412	7930916	8000294	8068580	8135810
2	7723217	7795965	7867514	7937904	8007171	8075350	8142476
3	7730547	7803173	7874605	7944880	8014037	8082110	8149132
4	7737864	7810369	7881684	7951846	8020893	8088859	8155777
5	7745170	7817554	7888751	7958800	8027737	8095597	8162413
6	7752463	7824726	7895807	7965743	8034571	8102325	8169038
7	7759743	7831887	7902852	7972675	8041394	8109043	8175654
8	7767012	7839036	7909885	7979596	8048207	8115750	8182259
9	7774268	7846173	7916906	7986506	8055009	8122447	8188854

	660	670	680	690	700	710	720
0	8195439	8260748	8325089	8388491	8450980	8512583	8573325
1	8202015	8267225	8331471	8394780	8457180	8518696	8579353
2	8208580	8273693	8337844	8401061	8463371	8524800	8585372
3	8215135	8280151	8344207	8407332	8469553	8530895	8591383
4	8221681	8286599	8350561	8413595	8475727	8536982	8597386
5	8228216	8293038	8356906	8419848	8481891	8543060	8603380
6	8234742	8299467	8363241	8426092	8488047	8549130	8609366
7	8241258	8305887	8369567	8432328	8494194	8555192	8615344
8	8247765	8312297	8375884	8438554	8500333	8561244	8621314
9	8254261	8318698	8382192	8444772	8506462	8567289	8627275
	730	740	750	760	770	780	790
0	8633229	8692317	8750613	8808136	8864907	8920946	8976271
1	8639174	8698182	8756399	8813847	8870544	8926510	8981765
2	8645111	8704039	8762178	8819550	8876173	8932068	8987252
3	8651040	8709888	8767950	8825245	8881795	8937618	8992732
4	8656961	8715729	8773713	8830934	8887410	8943161	8998205
5	8662873	8721563	8779470	8836614	8893017	8948697	9003671
6	8668778	8727388	8785218	8842288	8898617	8954225	9009131
7	8674675	8733206	8790959	8847954	8904210	8959747	9014583
8	8680564	8739016	8796692	8853612	8909796	8965262	9020029
9	8686444	8744818	8802418	8859263	8915375	8970770	9025468
	800	810	820	830	840	850	860
0	9030900	9084850	9138139	9190781	9242793	9294189	9344985
1	9036325	9090209	9143432	9196010	9247960	9299296	9350032
2	9041744	9095560	9148718	9201233	9253121	9304396	9355073
3	9047155	9100905	9153998	9206450	9258276	9309490	9360108
4	9052561	9106244	9159272	9211661	9263424	9314579	9365137
5	9057959	9111576	9164539	9216865	9268567	9319661	9370161
6	9063350	9116902	9169800	9222063	9273704	9324738	9375179
7	9068735	9122221	9175055	9227255	9278834	9329808	9380191
8	9074114	9127533	9180303	9232440	9283959	9334873	9385197
9	9079485	9132839	9185545	9237620	9289077	9339932	9390198
	870	880	890	900	910	920	930
0	9395193	9444827	9493900	9542425	9590414	9637878	9684829
1	9400182	9449759	9498777	9547248	9595184	9642596	9689497
2	9405165	9454686	9503649	9552065	9599948	9647309	9694159
3	9410142	9459607	9508515	9556878	9604708	9652017	9698816
4	9415114	9464523	9513375	9561684	9609462	9656720	9703469
5	9420081	9469433	9518230	9566486	9614211	9661417	9708116
6	9425041	9474337	9523080	9571282	9618955	9666110	9712758
7	9429996	9479236	9527924	9576073	9623693	9670797	9717396
8	9434945	9484130	9532763	9580858	9628427	9675480	9722028
9	9439889	9489018	9537597	9585639	9633155	9680157	9726656
	940	950	960	970	980	990	1000
0	9731279	9777236	9822712	9867717	9912261	9956352	0000000
1	9735896	9781805	9827234	9872192	9916690	9960737	
2	9740509	9786369	9831751	9876663	9921115	9965117	
3	9745117	9790929	9836263	9881128	9925535	9969492	
4	9749720	9795484	9840770	9885590	9929951	9973864	
5	9754318	9800034	9845273	9890046	9934362	9978231	
6	9758911	9804579	9849771	9894498	9938769	9982593	
7	9763500	9809119	9854265	9898946	9943172	9986952	
8	9768083	9813655	9858754	9903389	9947569	9991305	
9	9772662	9818186	9863238	9907827	9951963	9995655	

LOGARITHM~~IC SINES~~S.

of Nat. Sines

$$y_x = y_0 + x \cdot \frac{D_1}{1} + x(x-1) \cdot \frac{D_2}{1 \cdot 2} + x(x-1)(x-2) \cdot \frac{D_3}{1 \cdot 2 \cdot 3}.$$

From 0'1° to 0'5° inclusive the characteristic is $\bar{3}$, from 0'6° to 5'7° it is $\bar{2}$, and after that $\bar{1}$.

	0°	1°	2°	3°	4°	5°	6°
'0	— infin.	2418553	5428192	7188002	8435845	9402960	0192346
'1	2418771	2832434	5639994	7330272	8542905	9488739	0263865
'2	5429065	3210269	5841933	7468015	8647376	9572843	0334212
'3	7189966	3557835	6034886	7601512	8749381	9655337	0403424
'4	8439338	3879622	6219616	7731014	8849031	9736280	0471538
'5	9408419	4179190	6396796	7856753	8946433	9815729	0538588
'6	0200207	4459409	6567017	7978941	9041685	9893737	0604604
'7	0869646	4722626	6730804	8097772	9134881	9970356	0669619
'8	1449532	4970784	6888625	8213425	9226105	0045634	0733663
'9	1961020	5205514	7040899	8326066	9315439	0119616	0796762
	7°	8°	9°	10°	11°	12°	13°
'0	0858915	1435553	1943324	2396702	2805988	3178789	3520880
'1	0920237	1489148	1990913	2439472	2844803	3214297	3553582
'2	0980662	1542076	2037974	2481811	2883260	3249505	3586027
'3	1040246	1594354	2084516	2523729	2921367	3284416	3618217
'4	1099010	1645998	2130552	2565233	2959129	3319035	3650158
'5	1156977	1697021	2176092	2606330	2996553	3353368	3681853
'6	1214167	1747439	2221147	2647030	3033644	3387418	3713304
'7	1270600	1797265	2265725	2687338	3070407	3421190	3744517
'8	1326297	1846512	2309838	2727263	3106849	3454688	3775493
'9	1381275	1895195	2353494	2766811	3142975	3487917	3806237
	14°	15°	16°	17°	18°	19°	20°
'0	3836752	4129962	4403381	4659353	4899821	5126419	5340517
'1	3867040	4158152	4429728	4684069	4923083	5148371	5361286
'2	3897106	4186148	4455904	4708631	4946205	5170198	5381943
'3	3926952	4213950	4481909	4733043	4969192	5191904	5402489
'4	3956581	4241563	4507747	4757304	4992045	5213488	5422926
'5	3985996	4268988	4533418	4781418	5014764	5234953	5443253
'6	4015201	4296228	4558926	4805385	5037353	5256298	5463472
'7	4044196	4323285	4584271	4829208	5059818	5277526	5483585
'8	4072987	4350161	4609456	4852888	5082141	5298638	5503592
'9	4101575	4376859	4634483	4876426	5104343	5319635	5523494

	21°	22°	23°	24°	25°	26°	27°
·0	5543292	5735754	5918780	6093133	6259483	6418420	6570468
·1	5562987	5754468	5936594	6110118	6275701	6433926	6585312
·2	5582579	5773088	5954322	6127023	6291845	6449365	6600093
·3	5602071	5791616	5971965	6143850	6307917	6464735	6614810
·4	5621462	5810052	5989523	6160599	6323916	6480038	6629464
·5	5640754	5828397	6006997	6177270	6339844	6495274	6644056
·6	5659948	5846651	6024388	6193864	6355699	6510444	6658586
·7	5679044	5864816	6041696	6210382	6371484	6525548	6673054
·8	5698043	5882892	6058923	6226824	6387199	6540586	6687461
·9	5716946	5900880	6076068	6243190	6402844	6555559	6701807
	28°	29°	30°	31°	32°	33°	34°
·0	6716093	6855712	6989700	7118393	7242097	7361088	7475617
·1	6730319	6869359	7002802	7130983	7254204	7372737	7486833
·2	6744485	6882949	7015852	7143524	7266264	7384343	7498007
·3	6758592	6896484	7028849	7156015	7278277	7395904	7509140
·4	6772640	6909964	7041795	7168458	7290244	7407421	7520231
·5	6786629	6923388	7054689	7180851	7302165	7418895	7531280
·6	6800560	6936758	7067531	7193196	7314040	7430325	7542288
·7	6814434	6950074	7080323	7205493	7325870	7441712	7553256
·8	6828250	6963336	7093063	7217742	7337654	7453056	7564182
·9	6842010	6976545	7105753	7229943	7349393	7464358	7575068
	35°	36°	37°	38°	39°	40°	41°
·0	7585913	7692187	7794630	7893420	7988718	8080675	8169429
·1	7596718	7702601	7804671	7903104	7998062	8089692	8178133
·2	7607483	7712976	7814675	7912754	8007372	8098678	8186807
·3	7618208	7723314	7824643	7922369	8016649	8107631	8195450
·4	7628894	7733614	7834575	7931949	8025894	8116554	8204063
·5	7639540	7743876	7844471	7941496	8035105	8125444	8212646
·6	7650147	7754101	7854332	7951008	8044284	8134303	8221198
·7	7660715	7764289	7864157	7960486	8053430	8143131	8229721
·8	7671244	7774439	7873946	7969930	8062544	8151928	8238213
·9	7681735	7784553	7883701	7979341	8071626	8160694	8246676
	42°	43°	44°	45°	46°	47°	48°
·0	8255109	8337833	8417713	8494850	8569341	8641275	8710735
·1	8263512	8345948	8425548	8502417	8576648	8648331	8717548
·2	8271887	8354033	8433356	8509957	8583929	8655362	8724337
·3	8280231	8362091	8441137	8517471	8591186	8662369	8731102
·4	8288547	8370121	8448891	8524959	8598416	8669351	8737844
·5	8296833	8378122	8456618	8532421	8605622	8676309	8744561
·6	8305091	8386096	8464318	8539856	8612803	8683242	8751256
·7	8313320	8394041	8471991	8547266	8619958	8690152	8757927
·8	8321519	8401959	8479637	8554650	8627088	8697037	8764574
·9	8329691	8409850	8487257	8562008	8634194	8703898	8771198
	49°	50°	51°	52°	53°	54°	55°
·0	8777799	8842540	8905026	8965321	9023486	9079576	9133645
·1	8784376	8848889	8911153	8971233	9029188	9085073	9138943
·2	8790930	8855215	8917258	8977123	9034868	9090550	9144221
·3	8797462	8861519	8923342	8982992	9040529	9096007	9149479
·4	8803970	8867801	8929404	8988840	9046168	9101444	9154718
·5	8810455	8874061	8935444	8994667	9051787	9106860	9159937
·6	8816918	8880298	8941462	9000472	9057386	9112257	9165137
·7	8823357	8886513	8947459	9006257	9062964	9117634	9170317
·8	8829774	8892706	8953435	9012021	9068522	9122991	9175478
·9	8836168	8898877	8959389	9017764	9074059	9128328	9180620

	56°	57°	58°	59°	60°	61°	62°
'0	9185742	9235914	9284205	9330656	9375306	9418193	9459349
'1	9190845	9240827	9288932	9335201	9379674	9422386	9463371
'2	9195929	9245721	9293641	9339729	9384024	9426561	9467376
'3	9200994	9250597	9298332	9344238	9388356	9430720	9471364
'4	9206039	9255454	9303004	9348730	9392671	9434861	9475335
'5	9211066	9260292	9307658	9353204	9396968	9438985	9479289
'6	9216073	9265112	9312294	9357660	9401248	9443092	9483227
'7	9221062	9269913	9316911	9362098	9405510	9447182	9487147
'8	9226032	9274695	9321511	9366519	9409755	9451255	9491051
'9	9230982	9279459	9326092	9370921	9413982	9455310	9494938
	63°	64°	65°	66°	67°	68°	69°
'0	9498809	9536602	9572757	9607302	9640261	9671659	9701517
'1	9502663	9540291	9576284	9610668	9643470	9674713	9704419
'2	9506500	9543963	9579794	9614020	9646665	9677753	9707306
'3	9510320	9547619	9583288	9617355	9649843	9680777	9710178
'4	9514124	9551259	9586767	9620674	9653006	9683786	9713035
'5	9517912	9554882	9590229	9623978	9656153	9686779	9715876
'6	9521683	9558490	9593675	9627266	9659285	9689757	9718703
'7	9525437	9562081	9597106	9630538	9662402	9692720	9721514
'8	9529175	9565656	9600520	9633795	9665503	9695668	9724310
'9	9532897	9569215	9603919	9637036	9668588	9698600	9727092
	70°	71°	72°	73°	74°	75°	76°
'0	9729858	9756701	9782063	9805963	9828416	9849438	9869041
'1	9732610	9759303	9784519	9808273	9830583	9851462	9870924
'2	9735346	9761891	9786960	9810569	9832735	9853471	9872793
'3	9738067	9764464	9789386	9812850	9834872	9855467	9874648
'4	9740774	9767022	9791798	9815117	9836996	9857449	9876488
'5	9743466	9769566	9794195	9817370	9839105	9859416	9878315
'6	9746142	9772095	9796578	9819608	9841200	9861369	9880128
'7	9748804	9774609	9798946	9821831	9843281	9863308	9881927
'8	9751451	9777108	9801299	9824041	9845347	9865233	9883712
'9	9754083	9779593	9803639	9826236	9847400	9867144	9885482
	77°	78°	79°	80°	81°	82°	83°
'0	9887239	9904044	9919466	9933515	9946199	9957528	9967507
'1	9888982	9905648	9920932	9934844	9947393	9958586	9968431
'2	9890711	9907239	9922385	9936160	9948573	9959631	9969342
'3	9892427	9908815	9923824	9937463	9949740	9960663	9970239
'4	9894128	9910378	9925250	9938752	9950893	9961681	9971122
'5	9895815	9911927	9926661	9940027	9952033	9962686	9971993
'6	9897489	9913462	9928059	9941289	9953159	9963677	9972850
'7	9899148	9914984	9929444	9942537	9954271	9964655	9973693
'8	9900794	9916492	9930814	9943771	9955370	9965619	9974523
'9	9902426	9917986	9932171	9944992	9956456	9966570	9975340
	84°	85°	86°	87°	88°	89°	90°
'0	9976143	9983442	9989408	9994044	9997354	9999338	1'0000000
'1	9976933	9984099	9989931	9994435	9997612	9999461	
'2	9977710	9984742	9990441	9994812	9997856	9999577	
'3	9978473	9985372	9990938	9995176	9998088	9999676	
'4	9979223	9985988	9991422	9995527	9998306	9999762	
'5	9979960	9986591	9991892	9995865	9998512	9999835	
'6	9980683	9987181	9992349	9996189	9998703	9999894	
'7	9981393	9987758	9992793	9996500	9998882	9999940	
'8	9982089	9988321	9993223	9996798	9999047	9999974	
'9	9982772	9988871	9993640	9997082	9999200	9999993	

LOGARITHMIC TANGENTS.

$$y_x = y_0 + x \cdot \frac{D_1}{1} + x(x-1) \cdot \frac{D_2}{1 \cdot 2} + x(x-1)(x-2) \cdot \frac{D_3}{1 \cdot 2 \cdot 3}.$$

From	0·1°	0·6°	5·8°	45°	84·3°	89·5°
To	0·5°	5·7°	44·9°	84·2°	89·4°	89·9°
Char.	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	1	2

	0°	1°	2°	3°	4°	5°	6°
·0	0000000	2419215	5430638	7193958	8446437	9419518	0216202
·1	2418778	2833234	5642912	7336631	8554034	9505967	0288524
·2	5429091	3211221	5845136	7474792	8659055	9590754	0359688
·3	7190026	3558953	6038386	7608719	8761623	9673944	0429731
·4	8439444	3880918	6223427	7738665	8861850	9755597	0498689
·5	9408584	4180679	6400931	7864861	8959842	9835769	0566595
·6	0200445	4461103	6571490	7987519	9055697	9914514	0633482
·7	0869970	4724538	6735628	8106834	9149509	9991883	0699381
·8	1449956	4972928	6893813	8222984	9241363	0067924	0764321
·9	1961556	5207902	7046465	8336134	9331340	0142682	0828331
	7°	**8°**	**9°**	**10°**	**11°**	**12°**	**13°**
·0	0891438	1478025	1997125	2463188	2886523	3274745	3633641
·1	0953667	1532692	2045922	2507301	2926817	3311872	3668100
·2	1015044	1586706	2094203	2550997	2966769	3348711	3702315
·3	1075591	1640083	2141980	2594285	3006383	3385267	3736291
·4	1135333	1692839	2189264	2637173	3045667	3421546	3770030
·5	1194291	1744988	2236065	2679669	3084626	3457552	3803537
·6	1252486	1796546	2282395	2721780	3123266	3493290	3836816
·7	1309937	1847525	2328262	2763514	3161592	3528763	3869869
·8	1366665	1897939	2373678	2804878	3199611	3563977	3902700
·9	1422689	1947802	2418650	2845878	3237327	3598935	3935313
	14°	**15°**	**16°**	**17°**	**18°**	**19°**	**20°**
·0	3967711	4280525	4574964	4853390	5117760	5369719	5610659
·1	3999896	4310753	4603492	4880430	5143490	5394287	5634194
·2	4031873	4340800	4631863	4907332	5169097	5418747	5657633
·3	4063644	4370670	4660078	4934097	5194583	5443100	5680975
·4	4095212	4400363	4688139	4960727	5219950	5467346	5704223
·5	4126581	4429883	4716048	4987223	5245199	5491487	5727377
·6	4157752	4459232	4743808	5013588	5270331	5515524	5750438
·7	4188729	4488413	4771421	5039822	5295347	5539459	5773407
·8	4219515	4517427	4798887	5065928	5320250	5563292	5796286
·9	4250113	4546276	4826210	5091907	5345040	5587025	5819074

	21°	22°	23°	24°	25°	26°	27°
'0	5841774	6064096	6278519	6485831	6686725	6881818	7071659
'1	5864386	6085880	6299558	6506199	6706486	6901030	7090374
'2	5886912	6107586	6320527	6526503	6726190	6920189	7109041
'3	5909351	6129214	6341426	6546744	6745836	6939298	7127662
'4	5931705	6150766	6362257	6566923	6765426	6958355	7146237
'5	5953975	6172243	6383019	6587041	6784961	6977363	7164767
'6	5976162	6193645	6403714	6607097	6804440	6996320	7183251
'7	5998267	6214973	6424342	6627093	6823865	7015227	7201690
'8	6020290	6236227	6444903	6647030	6843236	7034086	7220085
'9	6042233	6257409	6465400	6666907	6862553	7052897	7238436
	28°	29°	30°	31°	32°	33°	34°
'0	7256744	7437520	7614394	7787737	7957892	8125174	8289874
'1	7275008	7455376	7631881	7804891	7974745	8141755	8306213
'2	7293230	7473194	7649334	7822013	7991569	8158311	8322529
'3	7311410	7490974	7666751	7839104	8008365	8174842	8338823
'4	7329547	7508716	7684135	7856164	8025133	8191348	8355094
'5	7347644	7526420	7701485	7873193	8041873	8207829	8371343
'6	7365699	7544088	7718801	7890192	8058587	8224286	8387571
'7	7383714	7561718	7736084	7907161	8075273	8240719	8403776
'8	7401689	7579313	7753334	7024101	8091933	8257127	8419961
'9	7419624	7596871	7770552	7941011	8108566	8273513	8436125
	35°	36°	37°	38°	39°	40°	41°
'0	8452268	8612610	8771144	8928098	9083692	9238135	9391631
'1	8468390	8628541	8786907	8943715	9099185	9253524	9406936
'2	8484492	8644454	8802654	8959319	9114666	9268904	9422233
'3	8500575	8660350	8818386	8974910	9130137	9284274	9437524
'4	8516637	8676228	8834103	8990487	9145596	9299636	9452807
'5	8532680	8692089	8849805	9006052	9161045	9314989	9468084
'6	8548704	8707933	8865492	9021604	9176483	9330334	9483355
'7	8564708	8723760	8881165	9037144	9191911	9345670	9498619
'8	8580694	8739571	8896823	9052672	9207329	9360998	9513876
'9	8596661	8755365	8912468	9068188	9222737	9376318	9529128
	42°	43°	44°	45°	46°	47°	48°
'0	9544374	9696559	9848372	0000000	0151628	0303441	0455626
'1	9559615	9711754	9863540	0015160	0166798	0318640	0470872
'2	9574850	9726945	9878706	0030320	0181970	0333843	0486124
'3	9590080	9742133	9893871	0045480	0197144	0349049	0501381
'4	9605305	9757318	9909035	0060641	0212321	0364260	0516645
'5	9620525	9772500	9924197	0075803	0227500	0379475	0531916
'6	9635740	9787679	9939359	0090965	0242682	0394695	0547193
'7	9650951	9802856	9954520	0106129	0257867	0409920	0562476
'8	9666157	9818030	9969680	0121294	0273055	0425150	0577767
'9	9681360	9833202	9984840	0136460	0288246	0440385	0593064
	49°	50°	51°	52°	53°	54°	55°
'0	0608369	0761865	0916308	1071902	1228856	1387390	1547732
'1	0623682	0777263	0931812	1087532	1244635	1403339	1563875
'2	0639002	0792671	0947328	1103177	1260429	1419306	1580039
'3	0654330	0808089	0962856	1118835	1276240	1435292	1596224
'4	0669666	0823517	0978396	1134508	1292067	1451296	1612429
'5	0685011	0838955	0993948	1150195	1307911	1467320	1628657
'6	0700364	0854404	1009513	1165897	1323772	1483363	1644906
'7	0715726	0869863	1025090	1181614	1339650	1499425	1661177
'8	0731096	0885334	1040681	1197346	1355546	1515508	1677471
'9	0746476	0900815	1056285	1213093	1371459	1531610	1693787

	56°	57°	58°	59°	60°	61°	62°
'0	1710126	1874826	2042108	2212263	2385606	2562480	2743256
'1	1726487	1891434	2058989	2229448	2403129	2580376	2761564
'2	1742873	1908067	2075899	2246666	2420687	2598311	2779915
'3	1759281	1924727	2092839	2263916	2438282	2616286	2798310
'4	1775714	1941413	2109808	2281199	2455912	2634301	2816749
'5	1792171	1958127	2126807	2298515	2473580	2652356	2835233
'6	1808652	1974867	2143836	2315865	2491284	2670453	2853763
'7	1825158	1991635	2160896	2333249	2509026	2688590	2872338
'8	1841689	2008431	2177987	2350666	2526806	2706770	2890959
'9	1858245	2025255	2195109	2368119	2544624	2724992	2909626
	63°	64°	65°	66°	67°	68°	69°
'0	2928341	3118182	3313275	3514169	3721481	3935904	4158226
'1	2947103	3137447	3333093	3534600	3742591	3957767	4180926
'2	2965914	3156764	3352970	3555097	3763773	3979710	4203714
'3	2984773	3176135	3372907	3575658	3785027	4001733	4226593
'4	3003680	3195560	3392903	3596286	3806355	4023838	4249562
'5	3022637	3215039	3412959	3616981	3827757	4046025	4272623
'6	3041645	3234574	3433077	3637743	3849234	4068295	4295777
'7	3060702	3254164	3453256	3658574	3870786	4090649	4319025
'8	3079811	3273810	3473497	3679473	3892414	4113088	4342367
'9	3098970	3293514	3493801	3700442	3914120	4135614	4365806
	70°	71°	72°	73°	74°	75°	76°
'0	4389341	4630281	4882240	5146610	5425036	5719475	6032289
'1	4412975	4654960	4908093	5173790	5453724	5749887	6064687
'2	4436708	4679750	4934072	5201113	5482573	5780485	6097300
'3	4460541	4704653	4960178	5228579	5511587	5811271	6130131
'4	4484476	4729669	4986412	5256192	5540768	5842248	6163184
'5	4508513	4754801	5012777	5283952	5570117	5873419	6196463
'6	4532654	4780050	5039273	5311861	5599637	5904788	6229970
'7	4556900	4805417	5065903	5339922	5629330	5936356	6263709
'8	4581253	4830903	5092668	5368137	5659200	5968127	6297685
'9	4605713	4856510	5119570	5396508	5689247	6000104	6331900
	77°	78°	79°	80°	81°	82°	83°
'0	6366359	6725255	7113477	7536812	8002875	8521975	9108562
'1	6401065	6762673	7154122	7581350	8052198	8577311	9171669
'2	6436023	6800389	7195122	7626322	8102061	8633335	9235679
'3	6471237	6838408	7236486	7671738	8152475	8690063	9300619
'4	6506710	6876734	7278220	7717605	8203454	8747514	9366518
'5	6542448	6915374	7320331	7763935	8255012	8805709	9433405
'6	6578454	6954333	7362827	7810736	8307161	8864667	9501311
'7	6614733	6993617	7405715	7858020	8359917	8924409	9570269
'8	6651289	7033231	7449003	7905797	8413294	8984956	9640312
'9	6688128	7073183	7492699	7954078	8467308	9046333	9711476
	84°	85°	86°	87°	88°	89°	90°
'0	9783798	0580482	1553563	2806042	4569162	7580785	+ infin.
'1	9857318	0668660	1663866	2953535	4792098	8038444	
'2	9932076	0758637	1777016	3106187	5027072	8550044	
'3	0008117	0850491	1893166	3264372	5275462	9130030	
'4	0085486	0944303	2012481	3428510	5538897	9799555	
'5	0164231	1040158	2135139	3599069	5819321	0591416	
'6	0244403	1138150	2261335	3776573	6119082	1560556	
'7	0326056	1238377	2391281	3961614	6441047	2809974	
'8	0409246	1340945	2525208	4154864	6788779	4570909	
'9	0494033	1445966	2663369	4357088	7166766	7581222	